

EXTENDED HYPERBOLICITY

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ABSTRACT. Given a complex space X , we considered the problem of finding a *hyperbolic model* of X . This is an object $\mathfrak{H}p(X)$ with a morphism $i : X \rightarrow \mathfrak{H}p(X)$ in such a way that $\mathfrak{H}p(X)$ is “hyperbolic” in a suitable sense and i is as close as possible to be an isomorphism. Using the theory of model categories, we found a definition of hyperbolic simplicial sheaf (for the strong topology) that extends the classical one of Brody for complex spaces. We prove the existence of hyperbolic models for any simplicial sheaf. Furthermore, the morphism i can be taken to be a cofibration and an affine weak equivalence (in an algebraic setting, Morel and Voevodsky called it an \mathbb{A}^1 weak equivalence). Imitating one possible definition of homotopy groups for a topological space, we defined the *holotopy* groups for a simplicial sheaf and showed that their vanishing in “positive” degrees is a necessary condition for a sheaf to be hyperbolic. We deduce that if X is a complex space with a non zero holotopy group in positive degree, then its hyperbolic model (that in general will only be a simplicial sheaf) cannot be weakly equivalent to a hyperbolic complex space (in particular is not itself hyperbolic). We finish the manuscript by applying these results and a *topological realization functor*, constructed in the previous section, to prove that the hyperbolic models of the complex projective spaces cannot be weakly equivalent to hyperbolic complex spaces.

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1. INTRODUCTION

The notion of hyperbolic space was given by Kobayashi in [9]. It is based on the existence of certain intrinsic distances, originally introduced to generalize Schwarz Lemma to higher dimensional complex spaces. Let $D \subset \mathbb{C}$ be the unit disc endowed with the Poincaré distance ρ . In view of the Schwarz-Pick Lemma every holomorphic map $f : D \rightarrow D$ is a contraction for ρ . Let X be a complex space. A *chain of holomorphic discs* between two points p, q in X is a set $\alpha = \{f_1, \dots, f_k\}$ of holomorphic maps $D \rightarrow X$ such that there are points $p = p_0, p_1, \dots, p_k = q$ in X and $a_1, b_1, \dots, a_k, b_k$ in D with the property $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$, $i = 1, \dots, k$. The *length* of α is defined as

$$(1) \quad l(\alpha) = \sum_{i=1}^k \rho(a_i, b_i)$$

The *Kobayashi pseudo distance* d_{Kob} on X is defined as

$$(2) \quad d_{Kob}^X(p, q) = \inf_{\alpha} l(\alpha)$$

where α varies through the family of all chains of holomorphic discs joining p and q . For quasi-projective varieties the pseudo distance of Kobayashi can be defined by means of chains of algebraic curves (see [6]).

The contraction property holds with respect to the Kobayashi pseudodistances: if $f : X \rightarrow Y$ is a holomorphic map between complex spaces, we have

$$(3) \quad d_{Kob}^Y(f(p), f(q)) \leq d_{Kob}^X(p, q),$$

for every $p, q \in X$. In particular, d_{Kob}^Y is invariant by biholomorphisms. It follows that $d_{Kob}^D = \rho$.

We have $d_{Kob}^{\mathbb{C}} \equiv 0$, $d_{Kob}^{\mathbb{C}^*} \equiv 0$. More generally, one has $d_{Kob}^G \equiv 0$ for every connected complex Lie group G (see [10]).

A complex space X is said to be *hyperbolic* (in the sense of Kobayashi) if d_{Kob}^X is a distance. The unit disc D is hyperbolic, whereas \mathbb{C} is not. $\mathbb{C} \setminus A$ with $\text{card } A \geq 2$ is hyperbolic. A compact complex curve of genus $g \geq 2$ is a hyperbolic space [10]. X is said to be *hyperbolic modulo C* , where C is a closed subset (usually a closed complex subspace), if for every pair of distinct points $x, y \in X \setminus C$ we have $d_{Kob}^X(x, y) > 0$.

If a complex space Y is \mathbb{C} -connected (i.e. for any $p \neq q$ points in Y there exists a holomorphic function $f : \mathbb{C} \rightarrow Y$ such that $p, q \in f(\mathbb{C})$) then, by virtue of the contraction property the only holomorphic maps with values in a hyperbolic space X are the constant ones. In particular, every holomorphic map $\mathbb{C} \rightarrow X$ is constant. The crucial fact is that for compact complex spaces the converse is also true. This is the content of the fundamental theorem of Brody (cfr. [10], [12]).

This result motivates the following definition: a complex space X is said to be *Brody hyperbolic* if every holomorphic map $f : \mathbb{C} \rightarrow X$ is constant.

As well as for hyperbolicity we have the notion of *Brody hyperbolicity modulo a closed subset C* : X is said to be *Brody hyperbolic modulo C* if every non constant holomorphic map $f : \mathbb{C} \rightarrow X$ satisfies $f(\mathbb{C}) \subset C$. A Kobayashi hyperbolic space is Brody hyperbolic but the converse is in general not true. Indeed Mark Green constructed a Zariski open set W in \mathbb{P}^2 (the two dimensional complex projective space), deleting four lines in general position and three points outside the four lines, which is Brody hyperbolic but not Kobayashi hyperbolic [12].

Related to hyperbolicity are some basic conjectures which motivated several important papers in Algebraic and Analytic Geometry.

- (1) A generic hypersurface of degree $\geq 2n + 1$ in \mathbb{P}^n is hyperbolic;
- (2) The complement of a hypersurface of degree $\leq 2n$ in \mathbb{P}^n is not hyperbolic;
- (3) A generic hypersurface of degree $\geq n + 2$ in \mathbb{P}^n is hyperbolic modulo a proper closed subvariety;

- (4) a smooth projective hyperbolic variety has an ample canonical bundle (Kobayashi's conjecture);
- (5) a smooth algebraic variety is of general type if and only if it is hyperbolic modulo a proper algebraic subset (Lang's conjecture [13], [14]).

For the basic material as well as a discussion of the geometric meaning of these conjectures we refer to [10], [12], [13], [14] and the bibliography there.

In this paper we will consider the following problem: given a complex space X , construct a "hyperbolic model" of X i.e. a "hyperbolic" object $\mathfrak{H}p(X)$, in a sense the "closest" hyperbolic object to X endowed with a canonical natural map $c_X : X \rightarrow \mathfrak{H}p(X)$ having the following universal property: if Y is hyperbolic a holomorphic map $f : X \rightarrow Y$ factorizes through $\mathfrak{H}p(X)$ i.e. we have a commutativity diagram

$$\begin{array}{ccc} X & \xrightarrow{c_X} & \mathfrak{H}p(X) \\ \downarrow f & \swarrow \tilde{f} & \\ Y & & \end{array}$$

One possible way to do this would be to consider the quotient topological space X/\mathcal{R} where \mathcal{R} is the equivalence relation: $x \sim y$ iff $d_{Kob}^X(x, y) = 0$ or, bearing in mind Brody's Theorem, if and only if they belong to the image of a holomorphic map $\mathbb{C} \rightarrow X$. This approach has two oddnesses. One is that X/\mathcal{R} is in general very different from X , indeed X/\mathcal{R} is just a point for \mathbb{C} -connected spaces X . On the other hand, X/\mathcal{R} will have in general no complex structure (even in a weak sense), thus it will be impossible to define a Kobayashi pseudodistance on this quotient in order to have an useful concept of hyperbolicity on it.

Regarding this, it is worth mentioning the nice paper of Campana [4] where a concept of Kobayashi pseudodistance is defined for orbifolds. Then, for any variety which is smooth and bimeromorphically equivalent to a Kähler manifold he constructed an orbifold $C(X)$ called the *core* of X and a meromorphic function $c_X : X \rightarrow C(X)$. Furthermore, he conjectured that the generic fiber of c_X has a vanishing Kobayashi metric and $C(X)$ is Brody hyperbolic modulo a proper subvariety.

In this paper we developed a different approach. We used techniques pioneered by Quillen in [17] and largely employed in [16], which we drew inspiration from in writing the technical sections of this paper. We construct an (unstable) homotopy category of complex spaces \mathcal{H} , whose objects include (homotopy) classes of complex spaces. Unlike the classical homotopy category of topological spaces, the category \mathcal{H} reflects the complex structure of the objects. The procedure involves an enlargement of the category of complex spaces to a new category containing as full subcategory the one of complex spaces with holomorphic functions. In this bigger category we define a notion of hyperbolicity which we prove that it restricts to the Brody hyperbolicity for complex spaces. Using this notion, we show that in each class of complex spaces lies a hyperbolic representative $\mathfrak{I}\mathfrak{p}(X)$, which in general will not be a complex space. It follows that $\mathfrak{I}\mathfrak{p}$ will be (weakly equivalent to) a point if and only if X is. Such correspondence is functorial and there exists a canonical morphism $c_X : X \rightarrow \mathfrak{I}\mathfrak{p}(X)$ satisfying the universality property described above. c_X and \tilde{f} will be morphisms of the homotopy category in general, but the composition $\tilde{f} \circ c_X$ is a class represented by a holomorphic function and the commutativity is as holomorphic functions as opposed to "up to homotopy". Concerning the object $\mathfrak{I}\mathfrak{p}(X)$, we will prove that the class of \mathbb{P}^n cannot have a hyperbolic complex space as representative, whereas in the class of \mathbb{C} , the point can be taken as hyperbolic complex space representative. $\mathfrak{I}\mathfrak{p}(X)$ is given by a complicated construction even if X is a complex space, although its *topological realization* (see Section 6) is a topological space homotopic equivalent to the topological space underlying X .

The procedure to construct the category \mathcal{H} follows closely the one described in [16] which works in a quite general context. It follows that almost all the results proved here are valid for algebraic schemes of finite type over a noetherian base, as well.

The main idea is to construct a category obtained from another by "adding" the inverses of certain morphisms. In the case of the category \mathbf{Compl} of complex

spaces with the strongly topology and holomorphic maps, we wish to add the inverse to the canonical map $p : \mathbb{C} \rightarrow \mathbf{pt}$ (the canonical projection $\mathbb{A}_B^1 \rightarrow B$ in the algebraic case) along with all its base changed maps. Such a category, which we denote as $p^{-1}\mathbf{CompI}$, exists, however, to make it usable, it should be obtained as the homotopy category associated to a *model structure* (see [17]) on \mathbf{CompI} . In general, deciding whether a localized category $S^{-1}\mathcal{C}$ is equivalent to the homotopy category associated to a model structure on \mathcal{C} is a very complicated task. This has been proved in the case of derived categories and the homotopy category of topological spaces. There are only partial results on this issue, if we assume that the category \mathcal{C} is a homotopy category itself and possesses a “simplicial structure”. The easiest way to replace a category \mathcal{C} with one endowed of such simplicial structure is to consider $\Delta^{op}\mathcal{C}$, the category of simplicial objects in \mathcal{C} . Then, we may try to give to $\Delta^{op}\mathcal{C}$ a *simplicial* model structure. If all this is successful, the homotopy category associated to such simplicial model structure is a good candidate to start with for establishing whether we can localize with respect of some morphism by using an appropriate model structure. In our situation, the category \mathbf{CompI} is replaced with $\mathcal{F}_T(\mathcal{S})$, the category of sheaves over the site of complex spaces endowed with the strong topology T (in the algebraic case, this will denote the category of sheaves over the site of smooth schemes of finite type over a noetherian base endowed with a topology not finer than quasi compact flat topology). The reason for doing this lies mainly in the fact that not all diagrams admit colimits and the existing ones in \mathbf{CompI} often are unsuitable to do homotopy theory with (see Section 2.1 for more details on this). On the other hand, $\mathcal{F}_T(\mathcal{S})$ is complete and cocomplete and the colimits have a “suitable” shape. We then proceed with the program described above in order to invert $p : \mathbb{C} \rightarrow \mathbf{pt}$. We end up with the category \mathcal{H}_s which is defined as the homotopy category associated to the simplicial model structure on $\Delta^{op}\mathcal{F}_T(\mathcal{S})$. The morphism p in the category \mathcal{H}_s fits in the Bousfield framework [1], and lies inside the class of weak equivalences in an appropriate model structure on $\Delta^{op}\mathcal{F}_T(\mathcal{S})$. The associated homotopy category will be denoted by \mathcal{H} and sometimes by \mathcal{H}^{olo} .

when we wish to stress that we are in the holomorphic setting. Any object of the site represents a class in \mathcal{H} and in the case it is a complex space, its hyperbolic model \tilde{X} will be only a simplicial sheaf on \mathbf{Cmpl} . The notion of hyperbolicity for a simplicial sheaf \mathcal{X} is given in the Definition 2.4. In the particular case $\mathcal{X} = X$ is a compact complex space, in view of Theorem 3.1 and Brody's Theorem, we conclude that X is hyperbolic according to our definition if and only if it is Kobayashi hyperbolic (see Corollary 3.1). Thus, the Definition 2.4 is a generalization of the classical concept of hyperbolicity for complex spaces.

In the section 4 we associate certain sets to each object of \mathcal{H}^{olo} which have a natural group structure in positive simplicial degree. They are called *holotopy sets* or *groups* when applicable (see Definition 4.1). We prove that the vanishing of some of the holotopy groups of a complex space X is a necessary condition for the hyperbolic model $\mathfrak{Ip}(X)$ to be isomorphic in \mathcal{H}^{olo} to some hyperbolic complex space.

In the following section we construct an useful functor for explicit computations: the *topological realization functor*. To a simplicial sheaf it associates a topological space in such a way few reasonable properties are satisfied (cfr. Definition 5.1). In the last section, as an application of some of our results, we show that $\mathfrak{Ip}(\mathbb{P}^n)$ is not weakly equivalent to a Brody hyperbolic space for any $n > 0$ and that the same holds for any complex space whose universal covering is \mathbb{C}^N for some $N > 0$.

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2. BASIC CONSTRUCTIONS

In this paper with \mathbb{P}^n we will denote the n -th dimensional projective complex space. The general problem we are dealing with is to modify the category of complex spaces to a category where the constant morphism $\mathbf{p} : \mathbb{C} \rightarrow \mathbf{pt}$ is invertible. The task of inverting morphisms in a category, can be accomplished by starting from an arbitrary category \mathcal{C} with respect to a given family S of morphisms satisfying

suitable compatibility conditions (cfr. [7]). The category $S^{-1}\mathcal{C}$ that we obtain is called the *localization* of \mathcal{C} with respect to S . In this kind of generality, $S^{-1}\mathcal{C}$ is not practical to work with. In this sense, reasonable categories are the "homotopy categories" associated to a *model structure* in the sense of Quillen (i.e. endowed with a "good definition" of *weak equivalence* [17]).

In this section we recall the main results of [16]. The constructions made there hold in the general context of a site with enough points in the sense of [8]. We restrict ourselves to the site \mathcal{S}_T of complex spaces with the strong topology or that of schemes of finite type over a noetherian scheme B of finite dimension, endowed with a Grothendieck topology which is weaker or as fine as the quasi compact flat topology.

2.1. Sheaves and simplicial objects: the categories $\mathcal{F}_T(\mathcal{S})$ and $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$.

Let \mathcal{S} be the category of complex spaces or schemes of finite type over a noetherian scheme B . If we wish to do some kind of homotopy theory on it, we should check the shape of colimits of certain diagrams. Recall that given a diagram \mathbf{D}

$$(4) \quad \begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow f & & \\ B & & \end{array}$$

in a small category \mathcal{C} , an object $\text{colim}_{\mathbf{D}}$ in \mathcal{C} is the colimit of \mathbf{D} if and only if $\text{colim}_{\mathbf{D}}$ fits in the commutative diagram

$$(5) \quad \begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow f & & \downarrow p \\ B & \xrightarrow{g} & \text{colim}_{\mathbf{D}} \end{array}$$

and $\text{Hom}_{\mathcal{C}}(\text{colim}_{\mathbf{D}}, X)$ is the limit of the diagram

$$(6) \quad \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(A, Z) & \xleftarrow{i^*} & \text{Hom}_{\mathcal{C}}(X, Z) \\ \uparrow f^* & & \\ \text{Hom}_{\mathcal{C}}(B, Z) & & \end{array}$$

in the category of sets for any $Z \in \mathcal{C}$. In other words, this last condition means that $\text{Hom}_{\mathcal{C}}(\text{colim}_{\mathbf{D}}, X)$ are pairs of morphisms (α, β) , $\alpha \in \text{Hom}_{\mathcal{C}}(X, Z)$ and $\beta \in \text{Hom}_{\mathcal{C}}(B, Z)$ with the property that $i^*\alpha = f^*\beta$. The definition of colimit of an

arbitrary diagram is similarly reduced to the one of limit in the category of sets by applying $\text{Hom}_{\mathcal{C}}(_, Z)$. We are particularly interested in colimits of diagrams of the kind

$$(7) \quad \begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow f & & \\ \text{pt} & & \end{array}$$

where i is an injection. In this paper, such colimits will sometimes be called *quotient* of X by A along i . In general, it may happen that the quotient does not exist in the category \mathcal{S} or if it exists, it is different from the one taken in the underlying category of topological spaces.

Examples 2.1. 1) Let \mathbf{D} be the diagrams

$$(8) \quad \begin{array}{ccc} \mathbb{C} - 0 & \xrightarrow{i} & \mathbb{C} \\ \downarrow f & & \downarrow f \\ \text{pt} & & \text{pt} \end{array} \quad \begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{i} & \mathbb{P}^2 \\ \downarrow f & & \downarrow f \\ \text{pt} & & \text{pt} \end{array}$$

where i are the canonical embeddings. Then the colimits of \mathbf{D} in \mathbf{Comp} are just a point in both cases, unlike their respective colimits in the category of topological spaces.

2) Let \mathbf{D} be the diagram

$$(9) \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{i} & \mathbb{C} \\ \downarrow f & & \\ \text{pt} & & \end{array}$$

where i is the canonical injection. \mathbf{D} has no colimit in \mathcal{S} . Indeed, by contradiction, let $Z = \text{colim}_{\mathbf{D}}$ in \mathcal{S} , $p : \mathbb{C} \rightarrow Z$ the corresponding canonical holomorphic function and $x = p(\mathbb{Z})$. Since there exists a non constant holomorphic function $h : \mathbb{C} \rightarrow \mathbb{C}$ such that $h(n) = 0$ for every $n \in \mathbb{Z}$, Z cannot be just the point x , moreover $p^{-1}(x) = i(\mathbb{Z})$. Let U be a relatively compact neighbourhood of x and $\{z^{(n)}\} \subset p^{-1}(U)$ a sequence with no accumulation points. If $h : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function satisfying $h(n) = 0$ and $h(z^{(n)}) = n$ for every $n \in \mathbb{Z}$, no holomorphic function $g : Z \rightarrow \mathbb{C}$ exists such that $g \circ p = h$.

A similar argument can be used to prove that the diagram

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{i} & \mathbb{C} \times \mathbb{C} \\ \downarrow f & & \\ \mathbf{pt} & & \end{array}$$

where i is the injection $\mathbb{C} \rightarrow \{0\} \times \mathbb{C}$, has no colimit in \mathcal{S} .

3) Let \mathbf{D} be the diagram

$$(11) \quad \begin{array}{ccc} \{0\} \cup \{1\} & \xrightarrow{i} & \mathbb{A}_k^1 \\ \downarrow f & & \\ \mathbf{pt} & & \end{array}$$

where \mathbb{A}_k^1 is the affine line over a field k and i is the embedding of the corresponding rational points. Then, since the k -algebra of the polynomials $P(x)$ of the form $a + x(x-1)Q(x)$, $a \in k$, is not finitely generated, \mathbf{D} has no colimit in the category of the algebraic schemes of finite type over k .

We therefore enlarge \mathcal{S} to a category which contains the colimits of all diagrams and, at the same time, have a “reasonably good” shape from our point of view. Such a category is $\mathcal{F}_T(\mathcal{S})$: the objects are sheaves of sets on a site \mathcal{S} endowed with the Grothendieck topology T and morphisms are maps of sheaves of sets. Recall that a *sheaf of sets* on \mathcal{S}_T (or an arbitrary site) is a contravariant functor $\mathcal{F} : \mathcal{S}_T \rightarrow \mathbf{Sets}$ satisfying the following conditions:

- (1) $\mathcal{F}(\emptyset) = \{\mathbf{pt}\}$, where \mathbf{pt} is the final object of \mathcal{S}_T ;
- (2) let $q : E \rightarrow X$ be a covering for the topology T , q_1 and q_2 respectively the canonical projections $E \times_X E \rightarrow E$; then

$$(12) \quad \mathcal{F}(X) \xrightarrow{q^*} \mathcal{F}(E) \begin{array}{c} \xrightarrow{q_1^*} \\ \xrightarrow{q_2^*} \end{array} \mathcal{F}(E \times_X E)$$

is an exact sequence of sets i.e.

$$q^* \mathcal{F}(X) = \{a \in \mathcal{F}(E) : q_1^*(a) = q_2^*(a)\}$$

Let $\mathbf{Y}(X) := \mathbf{Hom}_{\mathcal{S}}(\cdot, X)$. The functorial equality

$$\mathbf{Hom}_{\mathcal{S}}(A, B) = \mathbf{Hom}_{\mathbf{Funt}(\mathcal{S}^{op}, \mathbf{Sets})}(\mathbf{Y}(A), \mathbf{Y}(B))$$

is known as Yoneda Lemma. The Yoneda embedding is a faithfully full functor $\mathbf{Y} : \mathcal{S} \hookrightarrow \text{Fun}(\mathcal{S}^{op}, \text{Sets})$. If the topology T is not finer than the quasi compact flat topology, then the image of \mathbf{Y} is contained in the full subcategory $\mathcal{F}_T(\mathcal{S})$.

Theorem 2.1. *Let $X \in \mathcal{S}_T$ and T a topology not finer than the quasi compact flat topology or the strong topology in the holomorphic case. Then the functor $\text{Hom}_{\mathcal{S}}(\cdot, X)$ is a sheaf for the topology T .*

Proof. In the algebraic case, we restrict the proof to the case in which \mathcal{S}_T is a site of shemes of finite type over a base as we have been assuming from the very beginning. Then the conclusion follows from the theorem of Amitsur [15].

Assume now that \mathcal{S}_T is the site of complex spaces and let $q : E \rightarrow Z$ be an open covering of the complex space Z . Then, the sequence

$$(13) \quad E \times_Z E \begin{matrix} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{matrix} E \xrightarrow{q} Z$$

is exact as sequence of sets. We have to prove that the sequence of sets

$$(14) \quad \text{Hom}_{\mathcal{S}}(E \times_Z E, X) \begin{matrix} \xrightarrow{q_1^*} \\ \xrightarrow{q_2^*} \end{matrix} \text{Hom}_{\mathcal{S}}(E, X) \xleftarrow{q^*} \text{Hom}_{\mathcal{S}}(Z, X)$$

is exact, as well. Suppose that $q_1^* f = q_2^* f$ with $f \in \text{Hom}_{\mathcal{S}}(E, X)$. Since q is continuous, surjective and Z has the quotient topology induced by q , applying the functor $\text{Hom}_{\text{Top}}(\cdot, X)$ to the exact sequence (13) we obtain an exact sequence, hence a continuous map $f' : Z \rightarrow X$ such that $f = f' \circ q$. It follows that f' is holomorphic, f being holomorphic and q a local biholomorphism. \square

The category $\mathcal{F}_T(\mathcal{S})$ is complete and cocomplete. Indeed, the limit of a diagram \mathbf{D} in $\mathcal{F}_T(\mathcal{S})$ is the functor $U \rightsquigarrow \lim \mathbf{D}(U)$ which is a sheaf for the topology T . As for the colimit, it is defined as $a_T(U \rightsquigarrow \text{colim} \mathbf{D}(U))$ where a_T is the associated sheaf. In particular it possesses two canonical objects: an initial sheaf \emptyset , the sheaf that associates the empty set to any element of the site, except for the initial object of the site \mathcal{S} to which it associates the one point set and the final sheaf, which we will denote as \mathbf{pt} or $\text{Spec } B$ if the objects of the site are complex spaces or schemes over B , respectively.

We now would like to consider the localized category $p^{-1}\mathcal{F}_T(\mathcal{S})$, where $p : \mathbb{C} \rightarrow \mathbf{pt}$ (or $p : \mathbb{A}^1 \rightarrow \mathbf{Spec} k$). Moreover, we wish the localized category to have supplementary structures such as the ones we would get if $p^{-1}\mathcal{F}_T(\mathcal{S})$ were equivalent to the homotopy category of an appropriate model structure on $\mathcal{F}_T(\mathcal{S})$. Basically, a model structure on a category \mathcal{C} is the data of three classes of morphisms: weak equivalences, cofibrations and fibrations satisfying five axioms CM1, \dots , CM5 (see [17]) with the request that, in addition, the factorizations of CM5 are functorial. We do not know about the existence of such model structure on $\mathcal{F}_T(\mathcal{S})$. This is a particular case of the more general and complicated question on whether a localized category $S^{-1}\mathcal{C}$ is equivalent to the homotopy category associated to some model structure on \mathcal{C} . Some results of this kind are known in the case \mathcal{C} itself is a homotopy category (see [1]). To use them, we are forced to embed $\mathcal{F}_T(\mathcal{S})$ in the “simplest” category we know that is endowed of a model structure, namely $\Delta^{op}\mathcal{F}_T(\mathcal{S})$, the category of simplicial objects in $\mathcal{F}_T(\mathcal{S})$.

A *simplicial object* \mathcal{X} in \mathcal{C} is a sequence $\{\mathcal{X}_i\}_{i \geq 0}$ of objects of \mathcal{C} with a sequence $\partial_i^n : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1}$ of morphisms for $n \geq 1, i = 0, 1, \dots, n$ called *faces* and a sequence $\sigma_i^n : \mathcal{X}_n \rightarrow \mathcal{X}_{n+1}$ of morphisms for $n \geq 0, i = 0, 1, \dots, n$ called *degenerations*, satisfying the following conditions

$$\begin{aligned} 1) \quad & \partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{if } i < j \\ 2) \quad & \sigma_i \sigma_j = \sigma_{j+1} \sigma_i \quad \text{if } i \leq j \\ 3) \quad & \partial_i \sigma_j = \begin{cases} \sigma_{j-1} \partial_i & \text{if } i < j \\ \text{identity} & \text{if } i = j \text{ or } i = j + 1 \\ \sigma_j \partial_{i-1} & \text{if } i > j + 1 \end{cases} \end{aligned}$$

A *morphism* $f : \mathcal{X} \rightarrow \mathcal{Y}$ of two simplicial objects $\mathcal{X} = \{\mathcal{X}_i\}_{i \geq 0}, \mathcal{Y} = \{\mathcal{Y}_i\}_{i \geq 0}$ of \mathcal{C} is a sequence $\{f_i\}_{i \geq 0}$ of morphisms $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$ which make the diagrams

$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow{f_i} & \mathcal{Y}_i \\ \downarrow \sigma_i^n & & \downarrow \sigma_i^n \\ \mathcal{X}_{i+1} & \xrightarrow{f_{i+1}} & \mathcal{Y}_{i+1} \end{array} \quad \begin{array}{ccc} \mathcal{X}_i & \xrightarrow{f_i} & \mathcal{Y}_i \\ \downarrow \partial_i^n & & \downarrow \partial_i^n \\ \mathcal{X}_{i-1} & \xrightarrow{f_{i-1}} & \mathcal{Y}_{i-1} \end{array}$$

commutative.

With this notion of morphism, the family of simplicial objects of \mathcal{C} forms a category denoted by $\Delta^{\text{op}}\mathcal{C}$. Given $X \in \mathcal{C}$ we denote by the same symbol the *constant simplicial object* defined by $\mathcal{X}_i = X$, $\partial_i^n = \sigma_i^n = \text{Id}_X$, for every i, n .

Suppose that \mathcal{C} has a final object $*$, direct products and direct coproducts. Let $[n]$ be the set $\{0, 1, \dots, n\}$. Then, for every integer $n \geq 0$, denote by $\Delta[n]$ the simplicial object that at the level m has as many copies of $*$ as nondecreasing monotone functions $[m] \rightarrow [n]$. The $m+1$ injective functions $[m-1] \rightarrow [m]$ induce the faces and the m surjective functions $[m] \rightarrow [m-1]$ induce the degeneracies of $\Delta[n]$. On each copy of $*$ they act as the identity morphism. Notice that in $\Delta[n]_n$ there is only one nondegenerate element, namely the one corresponding to the identity. For example, $\Delta[1]$ is described as $\Delta[1]_i = \coprod_{j=1}^{i+2} *$ for each $i \geq 0$ and of the three $*$ in degree 1, two of them are the degenerations of the $*$ in degree 0. The two $*$ in degree zero are the images through the face morphisms of the nondegenerate $*$ in degree 1.

Remark 2.1. For every simplicial object \mathcal{X} , the pair of $*$ in degree 0 defines two morphisms ϵ_0 and $\epsilon_1 : \mathcal{X} \rightarrow \mathcal{X} \times \Delta[1]$.

Let \mathcal{X}, \mathcal{Y} two objects of $\Delta^{\text{op}}\mathcal{C}$.

Definition 2.1. A homotopy between two morphisms $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism $H : \mathcal{X} \times \Delta[1] \rightarrow \mathcal{Y}$ such that $H \circ \epsilon_0 = f$, $H \circ \epsilon_1 = g$.

In particular, this definition gives a notion of homotopy for objects and morphisms of \mathcal{C} .

Examples 2.2. 1) Let

$$\Delta_{\text{top}}^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : 0 \leq t_i \leq 1, \sum_i t_i = 1\}.$$

The collection $\{\Delta_{\text{top}}^n\}_n$ forms a cosimplicial topological space $\Delta_{\text{top}}^\bullet$ with the standard coface morphisms ∂^i (inclusion of the face missing the vertex v_i) and codegenerations σ^i (projection from v_i on the corresponding face).

- 2) Let \mathcal{C} be the category of sets. An object $A_\bullet = \{A_i\}_{i \geq 0}$ of $\Delta^{\text{op}}\mathcal{C}$ is called a *simplicial set*. The *geometrical realization* of A_\bullet is the topological space

$$|A_\bullet| = \frac{\coprod_n A_n \times \Delta_{\text{top}}^n}{(\partial_i(a), t) \sim (a, \partial^i(t))}.$$

A morphism $\phi : A_\bullet \rightarrow B_\bullet$ of simplicial objects is said to be a *weak equivalence* if its topological realization $|\phi| : |A_\bullet| \rightarrow |B_\bullet|$ is a weak equivalence, i.e. the homomorphisms $|\phi|_* : \pi_k(|A|, a) \rightarrow \pi_k(|B|, |\phi|(a))$, between the homotopy groups are isomorphisms, for all $k > 0$ and a bijection for $k = 0$.

- 3) Let \mathbf{Top} be the category of topological spaces with continuous maps. Then the functor $\text{Sing} : \mathbf{Top} \rightarrow \Delta^{\text{op}}\mathbf{Ins}$, which associates to a topological space K the simplicial set $\text{Hom}_{\mathbf{Top}}(\Delta_{\text{top}}^\bullet, K)$ is a functor that is left adjoint to $A_\bullet \rightsquigarrow |A_\bullet|$. The pair of adjoint functors $(\text{Sing}, | \cdot |)$

$$(15) \quad \Delta^{\text{op}}\mathbf{Ins} \begin{array}{c} \xleftarrow{\quad} \xrightarrow{\quad} \\ \text{Sing} \end{array} \mathbf{Top}$$

sends simplicial homotopies in the sense of Definition 2.1 to homotopies of topological spaces and viceversa.

A simplicial object in $\mathcal{F}_T(\mathcal{S})$ is said to be a *simplicial sheaf*. For the time being, we will consider $\mathcal{F}_T(\mathcal{S})$ as the full subcategory of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$, identified with constant simplicial sheaves.

2.2. Simplicial localization. The following will endow $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ with a model stucture in the sense of Quillen:

Definition 2.2. *A morphism $f : \mathcal{G} \rightarrow \mathcal{F}$ of simplicial sheaves is a weak equivalence if for every point x of a complex space or a scheme over B , $f_x : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is a weak equivalence of simplicial sets (\mathcal{G}_x and \mathcal{F}_x being the respective stalks over x of \mathcal{F} and \mathcal{G}).*

An injective morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be a *simplicial cofibration*.

A *lifting* in a commutative square of morphisms

$$(16) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{q} & \mathcal{X} \\ \downarrow j & & \downarrow f \\ \mathcal{B} & \xrightarrow{r} & \mathcal{Y} \end{array}$$

is a morphism $h : \mathcal{B} \rightarrow \mathcal{X}$ which makes the diagram commutative. In such situation we say that j has the *left lifting property* with respect to f and f has the *right lifting property* with respect to j .

A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called a *fibration* if all diagrams (16) admit a lifting, for all *acyclic cofibrations* j (cofibration and weak equivalence simultaneously).

The classes of weak equivalences, cofibrations and fibrations give $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a structure of *simplicial model category* as shown in [11]. Under these assumptions, there exists a localization of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ with respect of the weak equivalences. In other words, there exists a category which we will denote by \mathcal{H}_s and a functor

$$l : \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{H}_s$$

which has the properties

- 1) if f is a weak equivalence, $l(f)$ is an isomorphism;
- 2) the property is universal, namely, if another category \mathcal{C} exists and it is endowed with a functor $l' : \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{C}$ with the same property as l , then there exists a unique functor $u : \mathcal{H}_s^{\text{olo}} \rightarrow \mathcal{C}$ such that $l' = u \circ l$.

An object \mathcal{X} of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$

- 1) is called *cofibrant* if $\emptyset \rightarrow \mathcal{X}$ is a cofibration;
- 2) is called *fibrant* if $\mathcal{X} \rightarrow \text{pt}$ is a fibration.

2.3. Notations.

- 1) We denote pt the simplicial constant sheaf defined as the associated sheaf to the the presheaf which associates to an object of \mathcal{S} the set consisting of one element. The *pointed category* associated to $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ is the category $\Delta_{\bullet}^{\text{op}}\mathcal{F}_T(\mathcal{S})$ whose objects are the pairs (\mathcal{X}, x) where $\mathcal{X} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ and $x : \text{pt} \rightarrow \mathcal{X}$ is a morphism; a morphism of pairs $(\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ is a

morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that $f \circ x = y$. As pointed sheaf, \mathbf{pt} will stand for $(\mathbf{pt}, \mathbf{pt})$.

There is a pair of adjoint functors

$$(17) \quad \Delta^{\text{op}} \mathcal{F}_T(\mathcal{S}) \xrightleftharpoons[\mathbf{t}]{+} \Delta_{\bullet}^{\text{op}} \mathcal{F}_T(\mathcal{S})$$

where \mathbf{t} is the forgetful functor and $+$ is defined by $\mathcal{X} \rightsquigarrow \mathcal{X}_+$ with $\mathcal{X}_+ := \mathcal{X} \amalg \mathbf{pt}$, pointed by \mathbf{pt} .

- 2) Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of (pointed) simplicial sheaves. The symbol $\text{cof}(f)$ denotes the colimit of the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \\ \mathbf{pt} & & \end{array}$$

(pointed by the image of \mathcal{Y}) where \mathbf{pt} is a point. $\text{cof}(f)$ is called the *cofibre* of f . If f is a cofibration the cofibre of f is sometimes denoted by \mathcal{X}/\mathcal{Y} .

- 3) Let \mathcal{X} and \mathcal{Y} be pointed simplicial sheaves. The sheaf $\mathcal{X} \vee \mathcal{Y}$ is, by definition, the colimit of

$$\begin{array}{ccc} \mathbf{pt} & \longrightarrow & \mathcal{X} \\ \downarrow & & \\ \mathcal{Y} & & \end{array}$$

pointed by the image of \mathbf{pt} .

- 4) The pointed simplicial sheaf $\mathcal{X} \wedge \mathcal{Y}$ is defined by $\mathcal{X} \times \mathcal{Y} / \mathcal{X} \vee \mathcal{Y}$.
- 5) The simplicial pointed constant sheaf S_s^1 is defined by $\Delta[1] / \partial\Delta[1]$ where $\partial\Delta[1]$ is the simplicial subsheaf of $\Delta[1]$ consisting in the union of the images of the face morphisms of $\Delta[1]$. For $n \in \mathbb{N}$ we set $S_s^n = S_s^1 \wedge \cdots \wedge S_s^1$.

Remark 2.2. Performing the same constructions as for $\Delta^{\text{op}} \mathcal{F}_T(\mathcal{S})$ we obtain a homotopy category $\mathcal{H}_{s\bullet}$.

For a more complete description of the main properties of \mathcal{H}_s and $\mathcal{H}_{s\bullet}$ we refer to [17] e [16]. Here we only recall a proposition that will be used later.

Proposition 2.1. *Let $i : \mathcal{Y} \rightarrow \mathcal{X}$ be a simplicial cofibration of pointed simplicial sheaves. Then, for every pointed simplicial sheaf \mathcal{Z} , the morphism i induces a long exact sequence of pointed sets and groups (see the proof of Lemma 4.1)*

$$(18) \quad \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{Z}) \xleftarrow{i^*} \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Z}) \xleftarrow{\pi^*} \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}) \leftarrow \\ \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \xleftarrow{i^*} \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{X} \wedge S_s^1, \mathcal{Z}) \xleftarrow{\pi^*} \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{X}/\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \cdots$$

This proposition is a particular case of Proposition 4' of [17].

The Yoneda embedding, induces a functor $\mathbf{Y}_s : \mathcal{S}_T \rightarrow \mathcal{H}_s$ which is a full embedding (see the Proposition 1.13 and Remark 1.14 of [16]). However, in general, it is more difficult to describe the morphisms between objects in \mathcal{H}_s . Indeed, $\mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X})$ is obtained as a quotient of the set of diagrams

$$\mathcal{Y} \xrightarrow[\leftarrow]{\simeq} \mathcal{Y}' \rightarrow \mathcal{X}$$

of $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ where s is a weak equivalence.

\mathcal{H}_s (or its pointed version) is the appropriate category in which we are going to invert $p : \mathbb{C} \rightarrow \mathbf{pt}$. In the next section we will give a model structure to $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ whose weak equivalences contain p , and are in a sense the “smallest” class containing all the base changes of p as well. Such weak equivalences are written in terms of morphisms in \mathcal{H}_s and the homotopy category associated to this model structure is the localization of \mathcal{H}_s with respect to the weak equivalences.

2.4. Affine localization. Unless otherwise mentioned, the results presented in this subsection are taken from section 3.2 of [16].

Definition 2.3. *A simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ is said to be \mathbb{A}^1 -local (or \mathbb{C} local in the complex case) if the projection $\mathcal{Y} \times \mathbb{A}^1 \rightarrow \mathcal{Y}$ induces a bijection of sets*

$$\mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \rightarrow \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

for every $\mathcal{Y} \in \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$.

In what follows we describe a new structure of models on $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$, which we will call *affine*.

A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called:

- 1) an *affine* (or \mathbb{A}^1 in the algebraic case or \mathbb{C} in the complex case) *weak equivalence* if, for every \mathbb{A}^1 -local simplicial sheaf $\mathcal{Z} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$

$$f^* : \text{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Z}) \rightarrow \text{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{Z})$$

is a bijection;

- 2) an *affine cofibration* if it is injective;
- 3) an *affine fibration* if all diagrams (16) admit a lifting, where j is any affine cofibration and affine weak equivalence.

An object \mathcal{X} of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ is called

- 1) \mathbb{A}^1 -*fibrant* if the canonical morphism $X \rightarrow *$ is an affine fibration;
- 2) \mathbb{A}^1 -*cofibrant* if $\emptyset \rightarrow X$ an affine cofibration.

Theorem 2.2. (*cfr. Theorem 3.2, [16]*) *The structures listed above endow $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ of a model structure, which will be called affine model structure or \mathbb{A}^1 model structure.*

The localized category with respect of the affine weak equivalences is denoted as \mathcal{H} and its pointed version as \mathcal{H}_\bullet .

Remark 2.3. (1) Any object of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ is both (simplicially) cofibrant and \mathbb{A}^1 -cofibrant.

- (2) If $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a simplicial weak equivalence (respectively a simplicial cofibration) then it is an affine weak equivalence (respectively an affine cofibration). Therefore, the affine localization functor $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{H}$ factors as $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{H}_s \rightarrow \mathcal{H}$, where the first functor is the simplicial localization and the second is the identity on objects and identity on the fractions representing morphisms. However, the functor $\mathcal{H}_s \rightarrow \mathcal{H}$ is *not* an equivalence of categories.

- (3) The same classes of pointed morphisms, give $\Delta_\bullet^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a model structure.

Proposition 2.1 holds for \mathcal{H} as well

Proposition 2.2. *Let $j : \mathcal{Y} \rightarrow \mathcal{X}$ be an affine cofibration (i.e an injection of simplicial pointed sheaves). Then, for every simplicial pointed sheaf \mathcal{Z} , the morphism j induces long exact sequence of pointed sets and groups*

$$(19) \quad \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{Y}, \mathcal{Z}) \xleftarrow{j^*} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}, \mathcal{Z}) \xleftarrow{\pi^*} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}) \leftarrow \\ \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \xleftarrow{j^*} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{X} \wedge S_s^1) \xleftarrow{\pi^*} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathcal{X}/\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \cdots$$

The proof of such a statement is the same as for the Proposition 2.1.

2.5. Hyperbolic simplicial sheaves. Let us go back to the concept of hyperbolicity.

Definition 2.4. *A simplicial sheaf \mathcal{X} is said to be hyperbolic if it is \mathbb{A}^1 -local. Let \mathcal{C} be a simplicial subsheaf of \mathcal{X} . The simplicial sheaf \mathcal{X} is said to be hyperbolic mod \mathcal{C} if \mathcal{X}/\mathcal{C} is hyperbolic.*

Definition 2.5. *A hyperbolic resolution of \mathcal{X} is a morphism of simplicial sheaves $\mathfrak{r} : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ where $\tilde{\mathcal{X}}$ is a hyperbolic simplicial sheaf and \mathfrak{r} is an affine weak equivalence.*

A hyperbolic resolution functor is a pair $(\mathfrak{J}, \mathfrak{r})$ where \mathfrak{J} is a functor

$$\Delta^{\mathrm{op}} \mathcal{F}_T(\mathcal{S}) \rightarrow \Delta^{\mathrm{op}} \mathcal{F}_T(\mathcal{S})$$

and \mathfrak{r} is a natural transformation $\mathrm{Id} \rightarrow \mathfrak{J}$ such that every morphism $\mathcal{X} \rightarrow \mathfrak{J}(\mathcal{X})$ is a hyperbolic resolution.

From Proposition 2.19 of [16] we derive the following, fundamental result:

Theorem 2.3. *There exists a hyperbolic resolution functor $(\mathfrak{J}\mathfrak{p}, \mathfrak{r})$ with the following properties:*

- 1) *for every $\mathcal{X} \in \Delta^{\mathrm{op}} \mathcal{F}_T(\mathcal{S})$ the simplicial sheaf $\mathfrak{J}\mathfrak{p}(\mathcal{X})$ is hyperbolic and (simplicially) fibrant;*
- 2) *\mathfrak{r} is an affine equivalence and a cofibration;*
- 3) *let $\mathcal{H}_{s, \mathbb{A}^1}$ be the full subcategory in \mathcal{H}_s of \mathbb{A}^1 -local (hyperbolic) objects. $\mathfrak{J}\mathfrak{p}$ sends an affine weak equivalence to a simplicial weak equivalence, hence it*

- induces a functor $L : \mathcal{H}_s \rightarrow \mathcal{H}_{s, \mathbb{A}^1}$, that factors as $\mathcal{H}_s \rightarrow \mathcal{H} \rightarrow \mathcal{H}_{s, \mathbb{A}^1}$, where the first functor is the identity on objects (see also Remark 2.3 (2));
- 4) the canonical immersion $I : \mathcal{H}_{s, \mathbb{A}^1} \hookrightarrow \mathcal{H}_s$ is a right adjoint of L .

Furthermore, $\mathcal{H}_{s, \mathbb{A}^1}$ is a category equivalent to \mathcal{H} .

Given $X = \mathcal{X} \in \mathcal{F}_T$, $\mathfrak{I}\mathfrak{p}(X)$ is the hyperbolic simplicial sheaf associated to the simplicially constant sheaf X . However, due to its rather involved construction, the use of $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ is problematic even in the case when X is a complex space or a scheme over k .

Therefore, in general, the previous result shall be considered as an existence theorem. Nevertheless, it may occur that, in some particular cases, the class of $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ in \mathcal{H} could be represented by an understandable object, or even by a hyperbolic space (e.g. 3.1). In order to give a more precise idea of the difficulties involves, let $\underline{\text{Hom}}(\mathcal{X}, \mathcal{Z})$ be the right adjoint functor to $\mathcal{Y} \rightsquigarrow \mathcal{Y} \times \mathcal{X}$, where the objects are simplicial sheaves. Let us define $\text{Sing}_{\bullet}^{\mathbb{A}^1}(\mathcal{X})$ to be the simplicial sheaf $\{\underline{\text{Hom}}(\Delta_{\mathbb{A}^1}^n, \mathcal{X}_n)\}_{n \geq 0}$, where $\Delta_{\mathbb{A}^1}^{\bullet}$ is the cosimplicial sheaf such that $\Delta_{\mathbb{A}^1}^n = \mathbb{A}^n$ for every n and the structure morphisms are as described in page 88 of [16]. Then, the class $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ is defined to be the simplicial sheaf

$$(\text{Ex} \circ \text{Sing}_{\bullet}^{\mathbb{A}^1})^{\omega} \circ \text{Ex}(\mathcal{X})$$

where $\mathcal{X} \rightarrow \text{Ex}(\mathcal{X})$ is a fibrant simplicial resolution and ω is a sufficiently large ordinal.

We conclude this section by a short discussion on morphisms in localized categories. Morphisms in a localized category $S^{-1}\mathcal{C}$ can be expressed in terms of morphisms of \mathcal{C} using the so called *calculus of fractions*. More precisely,

$$\text{Hom}_{S^{-1}\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = \frac{\{\mathcal{X} \xleftarrow{s} \mathcal{X}' \xrightarrow{f} \mathcal{Y} : s \in S, f \in \text{Hom}_{\mathcal{C}}(\mathcal{X}', \mathcal{Y})\}}{\sim}$$

where the elements of the numerator are called *fractions* and \sim is an equivalence between fractions.

If the localization is associated to a model structure \mathcal{C} (as it happens for \mathcal{H}_s and \mathcal{H}), we know that there are objects \mathcal{X}, \mathcal{Y} such that, $\mathbf{Hom}_{S^{-1}\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ is a quotient of $\mathbf{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$; for instance, if \mathcal{X} is cofibrant and \mathcal{Y} is fibrant. Under these assumptions it can be proved that

$$(20) \quad \mathbf{Hom}_{S^{-1}\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = \frac{\mathbf{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})}{\sim_l}$$

where $f \sim_l g$ if and only if the morphism $f \amalg g$ factors through a *cylinder object* $\mathrm{Cyl}(\mathcal{X})$:

$$(21) \quad \begin{array}{ccc} & \mathrm{Cyl}(\mathcal{X}) & \\ \uparrow & \searrow \text{can} & \\ \mathcal{X} \amalg \mathcal{X} & \xrightarrow{f \amalg g} & \mathcal{Y} \end{array}$$

We recall that a cylinder associated to an object X of a category \mathcal{C} endowed with a model structure is an object $\mathrm{Cyl}(X) \in \mathcal{C}$ with morphisms

$$X \amalg X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{\text{can}} X$$

such that $\text{can} \circ i = \text{can} \circ \mathrm{id}_X \amalg \mathrm{id}_X$ and can is a weak equivalence. Cylinder objects always exists in a category \mathcal{C} endowed with a model structure. Furthermore the morphism i can be chosen to be a cofibration, can a fibration and the correspondence $X \rightsquigarrow \mathrm{Cyl}(X)$ functorial.

If $\mathcal{C} = \Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$, a cylinder object for the affine model structure associated to \mathcal{X} may be taken to be $\mathcal{X} \times \mathbb{A}^1$ where i is the morphism $X \amalg X \rightarrow \mathcal{X} \times \mathbb{A}^1$ determined by the inclusions at 0 and 1 (i.e. by the morphisms $\mathcal{X} \rightarrow \mathcal{X} \times \{0\}$, $\mathcal{X} \rightarrow \mathcal{X} \times \{1\}$) and can the projection onto \mathcal{X} . We already observed that every object of $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$ is cofibrant for both the model structures on $\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})$. Consequently, if \mathcal{Y} is fibrant (respectively simplicially fibrant), $\mathbf{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y})$ (respectively $\mathbf{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Y})$) is a quotient set of $\mathbf{Hom}_{\Delta^{\mathrm{op}}\mathcal{F}_T(\mathcal{S})}(\mathcal{X}, \mathcal{Y})$. In the sequel, this fact will be extensively used.

Lemma 2.1. *For any simplicial sheaf \mathcal{X} , the morphism $\mathfrak{r} : \mathcal{X} \rightarrow \mathfrak{Jp}(\mathcal{X})$ is universal in the category \mathcal{H} (respectively in the category \mathcal{H}_s) in the following sense: for any*

hyperbolic object \mathcal{Y} and morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ in \mathcal{H} , there exists a unique morphism $\tilde{f} : \mathfrak{I}\mathfrak{p}(\mathcal{Y}) \rightarrow \mathcal{Y}$ in \mathcal{H}_s .

Proof. Consider the commutative square

$$(22) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathfrak{r}_{\mathcal{X}}} & \mathfrak{I}\mathfrak{p}(\mathcal{X}) \\ \downarrow f & & \downarrow \mathfrak{I}\mathfrak{p}(f) \\ \mathcal{Y} & \xrightarrow[\cong]{\mathfrak{r}_{\mathcal{Y}}} & \mathfrak{I}\mathfrak{p}(\mathcal{Y}). \end{array}$$

By definition of \mathbb{A}^1 weak equivalence, $\mathfrak{r}_{\mathcal{Y}}$ is a simplicial weak equivalence, since both \mathcal{Y} and $\mathfrak{I}\mathfrak{p}(\mathcal{Y})$ are hyperbolic (i.e. \mathbb{A}^1 local). The map \tilde{f} is defined as $\mathfrak{r}_{\mathcal{Y}}^{-1} \circ \mathfrak{I}\mathfrak{p}(f)$. Note that $\mathfrak{I}\mathfrak{p}(f)$ is a morphism in \mathcal{H}_s . \square

Corollary 2.1. *Let X and Y be shaves with Y hyperbolic and $f : X \rightarrow Y$ be a morphism of sheaves. Then the composition $\tilde{f} \circ \mathfrak{r}$ is a morphism of sheaves and the commutativity of the diagram*

$$(23) \quad \begin{array}{ccc} X & \xrightarrow{\mathfrak{r}} & \mathfrak{I}\mathfrak{p}(X) \\ \downarrow f & \nearrow \tilde{f} & \\ Y & & \end{array}$$

is in the category of sheaves, i.e. it is strictly commutative and not only “up to homotopy” in \mathcal{H}_s .

Proof. By the previous lemma, we have commutativity in the category \mathcal{H}_s . Remark 1.14 of [16] implies that $\mathbf{Hom}(X, Y) = \mathbf{Hom}_{\mathcal{H}_s}(X, Y)$ since both X and Y have simplicial dimension zero. Therefore, equality of the morphisms f and $\tilde{f} \circ \mathfrak{r}$ in \mathcal{H}_s is an equality of morphisms of sheaves. \square

3. HYPERBOLICITY AND BRODY HYPERBOLICITY

In this section we will compare the different notions of hyperbolicity that we have introduced above. In particular, we prove that a simplicial sheaf represented by a complex space X is hyperbolic if and only if X is Brody hyperbolic. This is a corollary of the following

Theorem 3.1. *A sheaf $X \in \mathcal{F}_T(\mathcal{S})$ is a hyperbolic sheaf if and only if the projection $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection*

$$\mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, X)$$

for every object $U \in \mathcal{S}_T$. Moreover, under this hypothesis, for every $Y \in \mathcal{F}_T(\mathcal{S})$ there exists a bijection

$$(24) \quad \mathrm{Hom}_{\mathcal{H}}(Y, X) \cong \mathrm{Hom}_{\mathcal{H}_s}(Y, X) \cong \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X).$$

Remark 3.1. If in (24) the sets have a group structure induced (up to homotopy) by a group structure on Y or by a cogroup structure (up to homotopy) on X , the bijection is a group isomorphism.

Before beginning the proof, we fix, by the following commutative diagram

$$(25) \quad \begin{array}{ccccc} & & & & \mathcal{H}_s \\ & & \mathbf{Y}_s & \nearrow L_s & \uparrow I \\ S_T & \rightrightarrows & \mathcal{F}_T(\mathcal{S}) & \xrightarrow{\mathrm{cost}} & \Delta^{\mathrm{op}} \mathcal{F}_T(\mathcal{S}) \\ & & \mathbf{Y}_{\mathbb{A}^1} & \searrow L_{\mathbb{A}^1} & \downarrow L \\ & & & & \mathcal{H}_{s, \mathbb{A}^1} \cong \mathcal{H} \end{array}$$

the names of the functors involved in the proof. Notice that, the first functor on the left is the Yoneda embedding and L are the localization functors.

Proof of Theorem 3.1. First of all we have the following bijections of sets

$$(26) \quad \mathrm{Hom}_{\mathcal{H}}(\mathcal{Y}, \mathcal{X}) \cong \mathrm{Hom}_{\mathcal{H}}(L(\mathcal{Y}), \mathcal{X}) \cong \mathrm{Hom}_{\mathcal{H}_{s, \mathbb{A}^1}}(L(\mathcal{Y}), \mathcal{X}) \cong \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}).$$

The left end side bijection is a consequence of the fact that the canonical morphism $\mathcal{Y} \rightarrow L(\mathcal{Y})$ is an affine equivalence, the second one follows from the equivalence between \mathcal{H} and $\mathcal{H}_{s, \mathbb{A}^1}$ (Theorem 2.3) by definition of hyperbolicity of a simplicial sheaf. Finally, the third one follows from the fact that (L, I) is a pair of adjoint functors (Theorem 2.3.(4)).

Assume now that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are sheaves. Using the results quoted in [16, Remark 1.14, p. 52] one shows that

Lemma 3.1. *Let X, Y be sheaves. Then*

$$\mathrm{Hom}_{\mathcal{H}_s}(Y, X) \cong \mathrm{Hom}_{\mathcal{F}_T(S)}(Y, X).$$

This result implies the second assertion of Theorem 3.1. Indeed, if X is a hyperbolic sheaf, from Lemma 3.1 combined with the above considerations we get

$$\mathrm{Hom}_{\mathcal{H}}(Y, X) = \mathrm{Hom}_{\mathcal{H}_s}(Y, X) = \mathrm{Hom}_{\mathcal{F}_T(S)}(Y, X)$$

for every sheaf Y . Moreover, Lemma 3.1 also implies the first assertion in the following weaker form: given a sheaf X , the projection $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}_s}(U, X) \rightarrow \mathrm{Hom}_{\mathcal{H}_s}(U \times \mathbb{A}^1, X)$$

for every $U \in S_T$ if and only if it induces a bijection

$$\mathrm{Hom}_{\mathcal{F}_T(S)}(U, X) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(S)}(U \times \mathbb{A}^1, X).$$

Thus, in order to finish the proof of Theorem 3.1 in the general case, it is sufficient to prove the following: for every $U \in S_T$, the projection $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}_s}(U, X) \rightarrow \mathrm{Hom}_{\mathcal{H}_s}(U \times \mathbb{A}^1, X)$$

if and only if X is hyperbolic. For this we use the following Lemma 1.16 of [16]:

Lemma 3.2. *Let Σ a set of objects of $\mathcal{F}_T(S)$ such that, for every $U \in S_T$, there exists an epimorphism $F \rightarrow U$ where F is a direct sum of elements of Σ . Then there exist a functor*

$$\Phi_\Sigma : \Delta^{\mathrm{op}} \mathcal{F}_T(S) \rightarrow \Delta^{\mathrm{op}} \mathcal{F}_T(S)$$

and a natural transformation $\Phi_\Sigma \rightarrow \mathrm{Id}$ with the following properties: given \mathcal{Y}

- 1) *for every $n \geq 0$ the sheaf of sets $\Phi_\Sigma(\mathcal{Y})_n$ is a direct sum of sheaves belonging to Σ ;*
- 2) *the morphism $\Phi_\Sigma(\mathcal{Y}) \rightarrow \mathcal{Y}$ is both a (simplicial) weak equivalence and a local fibration (i.e. the morphism induces on the stalks a Kan fibration of simplicial sets).*

Since we will refer often to this lemma, we are going to recall here how to construct the functor Φ_Σ . Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of simplicial sheaves; define $\Psi_{\Sigma,f}$ as the colimit of

$$(27) \quad \begin{array}{ccc} \coprod_{\mathbf{D}_n, n \geq 0} F \times \partial\Delta[n] & \longrightarrow & \mathcal{X} \\ \downarrow & & \\ \coprod_{\mathbf{D}_n, n \geq 0} F \times \Delta[n] & & \end{array}$$

where \mathbf{D}_n is the set of the commutative squares of the kind

$$(28) \quad \begin{array}{ccc} F \times \partial\Delta[n] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ F \times \Delta[n] & \longrightarrow & \mathcal{Y} \end{array}$$

and $F \in \Sigma$. Let $\alpha_1 : \Psi_{\Sigma,f} \rightarrow \mathcal{Y}$ be the canonical morphism and $\Phi_{\Sigma,f}^{m+1}$ be Ψ_{Σ,α_m} . The $\Phi_{\Sigma,f}^i$ form a direct system of cofibrations $\{\Phi_{\Sigma,f}^1 \cdots \hookrightarrow \Phi_{\Sigma,f}^i \hookrightarrow \Phi_{\Sigma,f}^{i+1} \hookrightarrow \cdots\}$ whose colimit we will denote as $\Phi_{\Sigma,f}$. Such a simplicial sheaf factors functorially $f : \mathcal{X} \rightarrow \Phi_{\Sigma,f} \rightarrow \mathcal{Y}$. The functor that associates to a simplicial sheaf \mathcal{Y} the simplicial sheaf $\Phi_{\Sigma,\emptyset \rightarrow \mathcal{Y}}$ satisfies the properties of the Lemma.

In view of the Yoneda Lemma, we see that we can take as Σ the sheaves represented by objects in S_T . Indeed, for every sheaf W , we have a surjective morphism (even as presheaves)

$$(29) \quad \coprod_{U \in S_T} \coprod_{s \in W(U)} \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\cdot, U) \rightarrow W.$$

Thus, by Lemma 3.2, given an arbitrary simplicial sheaf \mathcal{Y} there exists \mathcal{Y}' such that $\mathcal{Y} = \mathcal{Y}'$ in \mathcal{H}_s and $\mathcal{Y}'_n = \coprod_{n_i} U_{n_i}$ with $U_{n_i} \in S_T$. By definition, if X is hyperbolic (i.e. \mathbb{A}^1 -local), the projection $\mathbf{p} : \mathcal{Y} \times \mathbb{A}^1 \rightarrow \mathcal{Y}$ induces a bijection

$$\mathbf{p}^* : \text{Hom}_{\mathcal{H}_s}(\mathcal{Y}, X) \rightarrow \text{Hom}_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{A}^1, X)$$

for every $\mathcal{Y} \in \mathcal{F}_T(\mathcal{S})$. In particular, this holds if $\mathcal{Y} = U \in \mathcal{S}_T$, so using Lemma 3.1 we conclude that for every $U \in \mathcal{F}_T(\mathcal{S})$

$$(30) \quad \mathbf{p}^* : \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \rightarrow \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, X)$$

is a bijection.

Conversely, assume that (30) is a bijection for every $U \in S_T$. Then, for every $U \in S_T$,

$$\mathbf{p}^* : \mathrm{Hom}_{\mathcal{H}_s}(U, X_+) \rightarrow \mathrm{Hom}_{\mathcal{H}_s}(U \times \mathbb{A}^1, X_+)$$

is a bijection. Let $sk_n \mathcal{Y}$ be the simplicial sheaf defined by

$$sk_n \mathcal{Y} = \begin{cases} (sk_n \mathcal{Y})_i = \mathcal{Y}_i, & \text{se } i \leq n, \\ (sk_n \mathcal{Y})_i = \Pi_{\sigma_{u_j} \text{ degenerations } \sigma_{u_1} \circ \cdots \circ \sigma_{u_{i-n}}(\mathcal{Y}_n)}, & \text{se } i > n. \end{cases}$$

Such an object is called the n -skeleton of \mathcal{Y} . The immersion

$$i_n : sk_{n-1} \mathcal{Y} \hookrightarrow sk_n \mathcal{Y}$$

is a cofibration and for $\mathcal{Y} = \mathcal{Y}'$, the cofibre $sk_n \mathcal{Y}' / sk_{n-1} \mathcal{Y}'$ is isomorphic to the sheaf $\Pi_{n_i} U_{n_i} \wedge S_s^n$.

We use the following cofibration sequences:

$$(31) \quad sk_{n-1} \mathcal{Y}'_+ \rightarrow sk_n \mathcal{Y}'_+ \rightarrow \vee_{n_i} U_{n_i+} \wedge S_s^n \rightarrow sk_{n-1} \mathcal{Y}'_+ \wedge S_s^1$$

$$(32) \quad \vee_n sk_n \mathcal{Y}'_+ \rightarrow \mathrm{dirlim}_n sk_n \mathcal{Y}'_+ = \mathcal{Y}'_+ \rightarrow \vee_n (sk_n \mathcal{Y}'_+ \wedge S_s^1) \xrightarrow{\mathrm{id} - \vee i_n \wedge \mathrm{id}_{S_s^1}} \vee_n (sk_n \mathcal{Y}'_+ \wedge S_s^1).$$

Notice that we are forced to take separate base points, since in the algebraic case, we cannot assume that a simplicial sheaf \mathcal{Z} can be considered as a pointed simplicial sheaf.

If $n = 1$ the sequence (31) becomes

$$(33) \quad (\Pi_{0_i} U_{0_i})_+ \hookrightarrow sk_1 \mathcal{Y}'_+ \rightarrow \vee_{1_i} U_{1_i+} \wedge S_s^1 \rightarrow (\Pi_{0_i} U_{0_i})_+ \wedge S_s^1.$$

Thus the following sequence

$$(34) \quad (\Pi_{0_i} U_{0_i} \times \mathbb{A}^1)_+ \hookrightarrow (sk_1 \mathcal{Y}' \times \mathbb{A}^1)_+ \rightarrow (\vee_{1_i} U_{1_i} \wedge S_s^1) \wedge \mathbb{A}_+^1 \rightarrow \Pi_{0_i} U_{0_i+} \wedge S_s^1 \wedge \mathbb{A}_+^1$$

is a cofibration sequence as well. The projection $\mathbf{p} : \mathbb{A}^1 \rightarrow \mathbf{pt}$ maps the latter sequence to the former.

Applying $\mathrm{Hom}_{\mathcal{H}_{s,\bullet}}(\cdot, X_+)$ we get the long exact sequence of pointed sets and groups

$$(35) \quad \text{Hom}_{\mathcal{H}_{s\bullet}}((\coprod_i U_{0_i})_+, X_+) \leftarrow \text{Hom}_{\mathcal{H}_{s\bullet}}(sk_1 \mathcal{Y}'_+, X_+) \leftarrow \\ \text{Hom}_{\mathcal{H}_{s\bullet}}(\vee_{1_i} U_{1_{i+}} \wedge S_s^1, X_+) \leftarrow \text{Hom}_{\mathcal{H}_{s\bullet}}((\coprod_i U_{0_i})_+ \wedge S_s^1, X_+) \cdots$$

as a particular case of the exact sequence (18). The morphism \mathbf{p}^* induces maps from the sequence (35) to the one corresponding to $\mathcal{Y} \times \mathbb{A}_+^1$. We are going to prove that \mathbf{p}^* is a bijection of pointed sets, from

$$\mathbf{A} = \text{Hom}_{\mathcal{H}_{s\bullet}}((\coprod_i U_{0_i})_+, X_+),$$

$$\mathbf{B} = \text{Hom}_{\mathcal{H}_{s\bullet}}(\vee_{1_i} U_{1_{i+}} \wedge S_s^1, X_+),$$

$$\mathbf{C} = \text{Hom}_{\mathcal{H}_{s\bullet}}(\coprod_i U_{0_{i+}} \wedge S_s^1, X_+).$$

\mathbf{p}^* is bijective from \mathbf{A} , because by the adjunction (17), we get

$$\mathbf{A} = \text{Hom}_{\mathcal{H}_s}(\coprod_i U_{0_i}, X_+).$$

Since direct sums of classes in \mathcal{H}_s are represented by direct sums in $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$, we have that $\mathbf{A} = \coprod_i \text{Hom}_{\mathcal{H}_s}(U_{0_i}, X_+)$ and we conclude by using the assumption we have on X .

Regarding the pointed set \mathbf{B} we argue as follows: a fibrant model of X_+ is of the kind \tilde{X}_+ , where \tilde{X} is a fibrant model of X , thus it is a nonconnected simplicial sheaf. On the other hand, $\vee_{1_i} U_{1_{i+}} \wedge S_s^1$ is a pointed connected simplicial sheaf. Since \mathbf{B} is a quotient set of $\text{Hom}_{\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})}(\vee_{1_i} U_{1_{i+}} \wedge S_s^1, \tilde{X}_+)$, we conclude that $\mathbf{B} = *$, the constant map to the base point, because $\text{Hom}_{\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})}(\vee_{1_i} U_{1_{i+}} \wedge S_s^1, \tilde{X}_+)$ is. The same argument works for

$$\text{Hom}_{\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})}(\vee_{1_i} U_{1_{i+}} \wedge S_s^1 \wedge (\mathbb{A}_+^1), \tilde{X}_+).$$

Thus \mathbf{p}^* is an isomorphism on \mathbf{B} . The same argument shows that \mathbf{p}^* is also an isomorphism on \mathbf{C} . By the Five Lemma, we conclude that \mathbf{p}^* is an isomorphism from $\text{Hom}_{\mathcal{H}_{s\bullet}}(sk_1 \mathcal{Y}'_+, X_+)$.

Similarly, we prove that the cofibration exact sequences (31) yield that \mathbf{p}^* is bijective from $\text{Hom}_{\mathcal{H}_{s\bullet}}(sk_n \mathcal{Y}'_+, X_+)$, for every $n \geq 0$.

Since \mathbf{p}^* is bijective from $\mathbf{C} = \mathrm{Hom}_{\mathcal{H}_s \bullet}((\coprod_i U_{0_i})_+ \wedge S_s^1, X_+)$ the same holds for $\mathrm{Hom}_{\mathcal{H}_s \bullet}(sk_1 \mathcal{Y}'_+ \wedge S_s^1, X_+)$ and consequently for $\mathrm{Hom}_{\mathcal{H}_s \bullet}(sk_n \mathcal{Y}'_+ \wedge S_s^1, X_+)$. Then, using the exactness of sequences (32), we conclude that \mathbf{p}^* is bijective from

$$\mathrm{Hom}_{\mathcal{H}_s \bullet}(\mathcal{Y}'_+, X_+) = \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}', X_+) = \mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, X_+),$$

thus from $\mathrm{Hom}_{\mathcal{H}_s}(\mathcal{Y}, X)$. Theorem 3.1 is completely proved. \square

Lemma 3.3. *Let $X \in \mathcal{S}_T$ and $\mathbf{p} : U \times \mathbb{A}^1 \rightarrow U$ be the projection. Then the map*

$$\mathbf{p}^* : \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{A}^1, X)$$

is bijective for every smooth scheme U if and only if

$$(36) \quad \mathbf{p}_{k(u)}^* : \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathrm{Spec} k(u), X) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{A}_{k(u)}^1, X)$$

is, for every finite field extension $\mathrm{Spec} L \rightarrow \mathrm{Spec} k$, $\mathbf{p}_L : \mathbb{A}_L^1 \rightarrow \mathrm{Spec} L$ being the projection.

Proof. We have just to prove that the bijectivity of \mathbf{p}_L^* for every L finite extension of k implies the bijectivity of \mathbf{p}^* for every smooth scheme U . The morphism $\mathbf{p} : U \times \mathbb{A}^1 \rightarrow U$ is a faithfully flat covering, thus, by faithfully flat descent we have the following exact sequence of sets

$$0 \rightarrow \mathrm{Hom}(U, X) \xrightarrow{\mathbf{p}^*} \mathrm{Hom}(U \times \mathbb{A}^1, X) \xrightarrow[\mathbf{p}_2^*]{\mathbf{p}_1^*} \mathrm{Hom}((U \times \mathbb{A}^1) \times_U (U \times \mathbb{A}^1), X).$$

In order to prove the surjectivity of \mathbf{p}^* , we have to show that $\mathbf{p}_1^* = \mathbf{p}_2^*$. Notice that

$$(U \times \mathbb{A}^1) \times_U (U \times \mathbb{A}^1) = U \times \mathbb{A}^2$$

and \mathbf{p}_1^* and \mathbf{p}_2^* are induced by the projections on the factors of \mathbb{A}^2 to \mathbb{A}^1 . Thus, given $\alpha \in \mathrm{Hom}(U \times \mathbb{A}^1, X)$, we prove that

$$\alpha \circ \mathbf{p}_1 = \alpha \circ \mathbf{p}_2 : U \times \mathbb{A}^2 \rightarrow X.$$

By hypothesis, any map $\mathbb{A}_L^1 \rightarrow X$ factors through $\mathrm{Spec} L$ for any finite extension L/k . In particular, $\alpha \circ \mathbf{p}_1$ and $\alpha \circ \mathbf{p}_2$ coincide on the closed points of $U \times \mathbb{A}^2$. Since the union of all closed points of $U \times \mathbb{A}^2$ is an everywhere dense subset for the Zariski topology, we conclude that $\alpha \circ \mathbf{p}_1 = \alpha \circ \mathbf{p}_2$. \square

Corollary 3.1. *Let X be a compact complex space. Then X is Kobayashi hyperbolic if and only if it is hyperbolic according to the definition 2.4.*

Proof. Consequence of Theorem 3.1, Lemma 3.3 and Brody's Theorem. \square

Corollary 3.2. *Let X be a complex space, C a closed complex subspace of X . Then X is hyperbolic modulo C in the sense of Brody if and only if X/C is a hyperbolic sheaf according to the definition 2.4.*

Proof. Let \mathcal{S}_T be the site of complex spaces. By definition, the sheaf of \mathcal{S}_T given by $Y \rightsquigarrow \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X/C)$ is the associated sheaf for the strong topology to the presheaf which associates to a complex space Y the colimit of

$$\begin{array}{ccc} \text{Hom}_{\mathcal{S}}(Y, C) & \longrightarrow & \text{Hom}_{\mathcal{S}}(Y, X) \\ \downarrow & & \\ \text{Hom}_{\mathcal{S}}(Y, \text{pt}). & & \end{array}$$

If X/C is a hyperbolic sheaf, then, by Theorem 3.1, we obtain that the morphism

$$\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\text{pt}, X/C) \rightarrow \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$$

is a bijection. Assume, by contradiction, that there exists a non constant holomorphic map $f : \mathbb{C} \rightarrow X$ such that $f(\mathbb{C}) \not\subset C$. Then f represents an element in $\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$ which is not in the image of $\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\text{pt}, X/C)$ which is absurd. Conversely, if X is Brody-hyperbolic modulo C , one has

$$\text{Hom}_{\mathcal{S}}(\mathbb{C}, X) = (X - C) \amalg \text{Hom}_{\mathcal{S}}(\mathbb{C}, C).$$

On the other hand, we observe that $\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}, X/C)$ is precisely equal to the colimit of

$$\begin{array}{ccc} \text{Hom}_{\mathcal{S}}(\mathbb{C}, C) & \longrightarrow & \text{Hom}_{\mathcal{S}}(\mathbb{C}, X) \\ \downarrow & & \\ \text{Hom}_{\mathcal{S}}(\mathbb{C}, \text{pt}). & & \end{array}$$

This follows from the fact that the new sections that we would get by taking the associated sheaf are of the form (f, g) where $f : U \rightarrow X, g : V \rightarrow X$ are holomorphic

maps, $\{U, V\}$ is an open covering of \mathbb{C} (we may assume both U and V to be connected) and $f(U \cap V), g(U \cap V)$ are contained in C . In this situation, we have that both $f(U)$ and $f(V)$ are contained in C , as well. Therefore, $(f, g) = (U \rightarrow \text{pt}, V \rightarrow \text{pt}) = \mathbb{C} \rightarrow \text{pt}$ and we already have this section in $\text{Hom}_{\mathcal{F}_T(S)}(\mathbb{C}, X/C)$. Consequently,

$$\text{Hom}_{\mathcal{F}_T(S)}(\mathbb{C}, X/C) = (X - C) \amalg \text{Hom}_S(\mathbb{C}, \text{pt})$$

but the latter set is $\text{Hom}_{\mathcal{F}_T(S)}(\text{pt}, X/C)$, thus X/C is a hyperbolic sheaf by Theorem 3.1. \square

Let us discuss some examples of hyperbolic resolutions of complex spaces. Roughly speaking, $\mathfrak{Jp}(X)$ "enlarges" X by adding a simplicial structure which trivializes passing from \mathcal{H}_s to \mathcal{H} . If X is a Brody hyperbolic complex space, $\mathfrak{Jp}(X)$ is isomorphic to X in the category \mathcal{H}_s . If X is not Brody hyperbolic the simplicial structures added to $\mathfrak{Jp}(X)$ have the task to "make constant" (up to simplicial homotopy, hence in \mathcal{H}_s) all morphisms $\mathbb{C} \rightarrow X$. Passing from \mathcal{H}_s to \mathcal{H} , X and $\mathfrak{Jp}(X)$ become isomorphic objects.

Examples 3.1. 1) $\mathfrak{Jp}(\mathbb{C})$ is a simplicial sheaf isomorphic to a point in \mathcal{H} .

Indeed, $\mathbb{C} \cong \text{pt}$ in \mathcal{H} and the hyperbolic resolutions preserve affine equivalences. This fact is not surprising because if we want to make all morphisms $\mathbb{C} \rightarrow \mathbb{C}$ homotopically constant, in particular this must be true for the identity $\mathbb{C} \rightarrow \mathbb{C}$.

2) For the same reason, $\mathfrak{Jp}(\mathbb{C}^n) \cong \text{pt}$ in \mathcal{H} for every $n \in \mathbb{N}$.

3) More generally, if $p : V \rightarrow X$ is a vector bundle, $\mathfrak{Jp}(p) : \mathfrak{Jp}(V) \xrightarrow{\cong} \mathfrak{Jp}(X)$ in \mathcal{H}_s because p is a \mathbb{C} weak equivalence. Therefore, if X is a hyperbolic complex space, then $\mathfrak{Jp}(V) \cong X$ in \mathcal{H}_s and hence in \mathcal{H} .

4) If X is a complex space and $\mathfrak{Jp}(X)$ is represented by a hyperbolic complex space Y , then Y is unique up to isomorphisms (cfr. the lemma below). In general, this is not the case; e.g. in the next section we will show that $\mathfrak{Jp}(\mathbb{P}^n)$ cannot be \mathbb{C} -equivalent to a hyperbolic complex space.

In the case $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ admits a hyperbolic complex space as representative, then such a space is unique up to biholomorphism:

Lemma 3.4. *Let \mathcal{X} be a simplicial sheaf, Y, Y' hyperbolic complex spaces such that*

$$\mathfrak{I}\mathfrak{p}(\mathcal{X}) = [Y]_{\mathcal{H}} = [Y']_{\mathcal{H}}.$$

Then Y' and Y are isomorphic complex spaces.

Proof. Let \mathcal{S} be the category of complex spaces. By hypothesis, there exists an isomorphism $\phi : Y \cong Y'$ in \mathcal{H} , namely a morphism $\psi : Y' \rightarrow Y$ in \mathcal{H} such that $\psi \circ \phi = \text{id}_Y$ and $\phi \circ \psi = \text{id}_{Y'}$ in \mathcal{H} . Since Y e Y' are complex hyperbolic spaces, and in particular \mathbb{C} -fibrant objects by Corollary 3.1 (see also the end of Section 2), ϕ and ψ can be represented by morphisms $\phi' : Y \rightarrow Y'$ and $\psi' : Y' \rightarrow Y$ in $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$. More precisely, we may suppose that ϕ' and ψ' are holomorphic maps, Y, Y' being complex spaces and $\mathcal{S} \hookrightarrow \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ being a full immersion. Moreover, the fact that ϕ, ψ are inverse to each other means that $\psi' \circ \phi' \sim \text{id}_Y$, $\phi' \circ \psi' \sim \text{id}_{Y'}$ as holomorphic maps, where $f \sim g$ if and only if there exists a holomorphic map $H : W \times \mathbb{C} \rightarrow V$ such that $H|_{W \times 0} = f$ e $H|_{W \times 1} = g$ (cfr. equation (20)). Since both Y, Y' are hyperbolic, H must be constant along the fibres which are isomorphic to \mathbb{C} , thus $f \sim g$ if and only if $f = g$ as maps. In particular, $\psi' \circ \phi' = \text{id}_Y$ and $\phi' \circ \psi' = \text{id}_{Y'}$. \square

In some cases, we can extend some results known for hyperbolic complex spaces to hyperbolic sheaves:

Lemma 3.5. *Let $\mathcal{F}_T(\mathcal{S})$ be the category of sheaves of sets on the site of complex spaces with the strong topology and F be a hyperbolic sheaf. Then*

$$\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{P}^n, F) = F(\text{pt})$$

for any $n \geq 1$. In other words, any sheaf map from \mathbb{P}^n to a hyperbolic sheaf F must be constant.

Proof. Consider the case $n = 1$ first. Let $\mathbb{P}^1 = U_0 \cup U_1$ be an open covering with $U_0 = \mathbb{P}^1 \setminus \{0\}$ and $U_1 = \mathbb{P}^1 \setminus \{\infty\}$. Then the square

$$(37) \quad \begin{array}{ccc} U_0 \cap U_1 & \xrightarrow{i_0} & U_0 \\ \downarrow i_1 & & \downarrow \\ U_1 & \xrightarrow{\quad} & \mathbb{P}^1 \end{array}$$

is cocartesian in the category of sheaves. Thus

$$(38) \quad \mathrm{Hom}_{\mathcal{F}_T(S)}(\mathbb{P}^1, F) = \lim \left(\begin{array}{ccc} \mathrm{Hom}_{\mathcal{F}_T(S)}(U_0 \cap U_1, F) & \xleftarrow{i_0^*} & \mathrm{Hom}_{\mathcal{F}_T(S)}(U_0, F) \\ & \uparrow i_1^* & \\ & \mathrm{Hom}_{\mathcal{F}_T(S)}(U_1, F) & \end{array} \right).$$

Since $U_0 \cong U_1 \cong \mathbb{C}$, we have that

$$\mathrm{Hom}_{\mathcal{F}_T(S)}(U_j, F) = \mathrm{Hom}_{\mathcal{F}_T(S)}(\mathrm{pt}, F) = F(\mathrm{pt})$$

for $j = 0, 1$ because of the theorem 3.1. Moreover, i_j^* are injective because they have a retraction given by f^* where $f : \mathrm{pt} \rightarrow U_0 \cap U_1$ is any point. We conclude the statement of the lemma in the case of \mathbb{P}^1 by noticing that the image of i_0^* coincides with the one of i_1^* . Consider now the open covering of \mathbb{P}^n given by $U_0 = \mathbb{P}^n \setminus \mathbb{P}^{n-1} \cong \mathbb{C}^n$ and $U_1 = \mathbb{P}^n \setminus \{\infty\}$, where ∞ coincides with the point $(0, 0, \dots, 0) \in U_0 = \mathbb{C}^n$. We get a cocartesian square like (37) with \mathbb{P}^n replacing \mathbb{P}^1 . The previous argument carries through in the general case. The only thing to check is that $\mathrm{Hom}_{\mathcal{F}_T(S)}(U_1, F) = F(\mathrm{pt})$. Notice that the canonical projection $p : U_1 \rightarrow \mathbb{P}^{n-1}$ is a rank one vector bundle. Locally on \mathbb{P}^{n-1} (for the strong topology) it is $V \times \mathbb{C}$, where V is an open affine of \mathbb{P}^{n-1} . Hence

$$(39) \quad p_V^* : \mathrm{Hom}_{\mathcal{F}_T(S)}(V, F) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(S)}(V \times \mathbb{C}, F)$$

are bijections for all V , since F is hyperbolic. Glueing these data for V ranging on an open affine covering of \mathbb{P}^{n-1} , we get that

$$(40) \quad p^* : \mathrm{Hom}_{\mathcal{F}_T(S)}(\mathbb{P}^{n-1}, F) \rightarrow \mathrm{Hom}_{\mathcal{F}_T(S)}(U_1, F)$$

is a bijection. By inductive assumption, we conclude that

$$\mathrm{Hom}_{\mathcal{F}_T(S)}(U_1, F) = F(\mathrm{pt}).$$

□

4. HOLOTOPY GROUPS

Throughout this section, S_T will denote the site of complex spaces endowed with the strong topology. A simplicial object of S_T is, by definition, a *simplicial complex space*. If we forget the complex structure, we could study the objects of S_T by means of the classical homotopy groups. Isomorphism classes of homotopy groups are invariant under homeomorphisms hence, a fortiori, under biholomorphisms, however, they do not reflect the existence and the properties of the complex structure. A rather natural modification of the definition of homotopy enables us to attach to every simplicial sheaf on S_T two families $\{\pi_{i,j}^{\text{par}}(\mathcal{X})\}_{i,j}$, $\{\pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X})\}_{m,n}$ of sets (cfr. Definition 4.1) which, for positive simplicial degrees, have a canonical group structure and are invariant under biholomorphisms. We will use these groups in Section 6 to show that there exist complex spaces (e.g \mathbb{P}^n) whose hyperbolic resolutions (cfr. Definition 2.5) are not isomorphic to the class of hyperbolic complex spaces, not even in the category \mathcal{H} .

Define the *parabolic circle* by

$$S_{\text{par}}^1 = \mathbb{C}/(0 \amalg 1),$$

and we denote by S_{par}^n the sheaf $S_{\text{par}}^1 \wedge \cdots \wedge S_{\text{par}}^1$.

Let $D \subset \mathbb{C}$ be the unit disc and $z_1 \neq z_2$ two points of D . We define the *hyperbolic circle* $S_{\text{iper}}^1(z_1, z_2)$ by

$$S_{\text{iper}}^1(z_1, z_2) = D/(z_1 \amalg z_2)$$

and we denote by $S_{\text{iper}}^n(z_1, z_2)$ the sheaf $S_{\text{iper}}^1(z_1, z_2) \wedge \cdots \wedge S_{\text{iper}}^1(z_1, z_2)$.

The quotients defining parabolic and hyperbolic circles are taken in the category $\mathcal{F}_T(\mathcal{S})$, even though, in view of a theorem of Cartan (cfr. [5]) the set theoretic quotients have a complex structure.

Definition 4.1. *Let \mathcal{X} be a simplicial sheaf on S_T . Define*

$$(41) \quad \pi_{i,j}^{\text{par}}(\mathcal{X}, x) = \text{Hom}_{\mathcal{H}_\bullet}((\mathbb{C} - 0)^{\wedge j} \wedge S_{\text{par}}^{i-j}, (\mathcal{X}, x))$$

for $i \geq j \geq 0$,

$$(42) \quad \pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X}, x) = \text{Hom}_{\mathcal{H}_\bullet}(S_{\text{iper}}^n(z_1, z_2) \wedge S_{\text{par}}^m, (\mathcal{X}, x))$$

for $n, m \geq 0$.

These sets are called respectively parabolic holotopy pointed sets of \mathcal{X} (or groups in the case they are) and hyperbolic holotopy pointed sets of \mathcal{X} (or groups in the case they are).

Remark 4.1. The definitions above are compatible with the classical ones of algebraic topology. More precisely, let \mathcal{H}^{top} be the (unstable) homotopy category of topological spaces (i.e. the localization of the category of topological spaces with respect to the usual weak equivalences); then we have $\pi_n(X, x) = \text{Hom}_{\mathcal{H}^{\text{top}}}((S^n, p), (X, x))$ for every topological space X . Moreover, the topological realization functor (cfr. Section 5) provides functorial group homomorphisms $\pi_{i,j}^{\text{par}}(X, x) \rightarrow \pi_{i-j}(X, x)$ and $\pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X}, x) \rightarrow \pi_m(X, x)$ for any complex space X .

Lemma 4.1. *The sets $\pi_{i,j}^{\text{par}}, \pi_{n,m}^{\text{iper}}$ have a canonical group structure for $i > j > 0$ and $m > 0$.*

Proof. The first step consists in proving that $S_{\text{par}}^1 \cong S_s^1$ in \mathcal{H} .

Consider the cofibration sequence

$$(43) \quad 0 \amalg 1 \hookrightarrow \mathbb{C} \rightarrow S_{\text{par}}^1 \rightarrow S_s^1 \rightarrow \mathbb{C} \wedge S_s^1 \rightarrow \dots$$

where $0 \amalg 1$ and \mathbb{C} are pointed by 0. Since $\mathbb{C} \cong \text{pt}$ in \mathcal{H} , we have $\mathbb{C} \wedge S_s^1 \cong \text{pt}$ in \mathcal{H} . Applying the functor $\text{Hom}_{\mathcal{H}_\bullet}(_, \mathcal{Z})$, in view of Proposition 2.2, we obtain long exact sequences of sets and, from these, the isomorphism

$$\text{Hom}_{\mathcal{H}_\bullet}(S_{\text{par}}^1, \mathcal{Z}) \cong \text{Hom}_{\mathcal{H}_\bullet}(S_s^1, \mathcal{Z})$$

for every $\mathcal{Z} \in \Delta_{\bullet}^{\text{op}} \mathcal{F}_T(S)$. It follows that $S_{\text{par}}^1 \cong S_s^1$ in \mathcal{H} . The simplicial object S_s^1 is a cogroup (object) in \mathcal{H}_s (and consequently in \mathcal{H}). It is sufficient to observe

that, if a_{str} is the associated sheaf for the strong topology, $S_s^1 \cong a_{\text{str}}(\text{Sing}(S^1))$ in \mathcal{H}_s and that S^1 is a cogroup in \mathcal{H}^{top} with projection

$$\mathbf{p} : S^1 \rightarrow S^1 / (\{i\} \amalg \{-i\}) \xrightarrow{\text{homeo}} S^1 \vee S^1$$

as structural map. Then, applying to \mathbf{p} the functor $a_{\text{str}}(\text{Sing}(\))$ we get a morphism

$$[S_s^1] \rightarrow [S_s^1 \vee S_s^1] = [S_s^1] \vee [S_s^1]$$

in \mathcal{H}_s which satisfies the properties making it a comultiplication. These properties are formulated in such a way to induce on the sets $\text{Hom}_{\mathcal{H}_s}(S_s^1, \mathcal{Z})$ a natural group structure. The same holds for $\text{Hom}_{\mathcal{H}}(S_s^1, \mathcal{Z})$. \square

Theorem 4.1. *Let X be a hyperbolic sheaf. Then the groups $\pi_{i,j}^{\text{par}}(X, x)$, $\pi_{n,m}^{\text{iper}}(X, x)$ vanish for $i - j > 0$ and any $m > 0$.*

Proof. We begin with proving that $\text{Hom}_{\mathcal{H}_\bullet}(Y \wedge S_{\text{par}}^1, X) = 0$ for every pointed complex space $(Y, \{y\})$. By definition (cfr. Section 2.3),

$$Y \wedge S_{\text{par}}^1 = Y \times \mathbb{C}/R$$

where R is the complex space $Y \times (0 \amalg 1) \cup y \times \mathbb{C}$. Since $Y \times \mathbb{C}/R$ is a sheaf and X is a fibrant space, by Theorem 3.1 we conclude that

$$\begin{aligned} (44) \quad \text{Hom}_{\mathcal{H}}(Y \times \mathbb{C}/R, X) &= \text{Hom}_{\mathcal{F}_T(S)}(Y \times \mathbb{C}/R, X) = \\ &= \{f \in \text{Hom}_{\mathcal{F}_T(S)}(Y \times \mathbb{C}, X) : f|_R = \text{constant}\}. \end{aligned}$$

Moreover, since $Y \times \mathbb{C}$ and X are complex spaces, we have

$$\text{Hom}_{\mathcal{F}_T(S)}(Y \times \mathbb{C}, X) = \text{Hom}_{\text{olom}}(Y \times \mathbb{C}, X).$$

X is Brody hyperbolic hence, for every $y \in Y$, the restriction of a holomorphic map $f : Y \times \mathbb{C} \rightarrow X$ to $y \times \mathbb{C}$ is constant. Furthermore, if $f \in \text{Hom}_{\mathcal{F}_T(S)}(Y \times \mathbb{C}/R, X)$ then f is constant on $Y \times 0 \subset R$ and consequently on the whole $Y \times \mathbb{C}$. It follows that, if f is pointed, then f must be constant with image x , the base point of X .

This shows that $\mathrm{Hom}_{\mathcal{H}_\bullet}(Y \wedge S_s^1, X) = x$ for any pointed complex space Y and any hyperbolic pointed sheaf X .

We would like now to prove the same result with Y being replaced by a quotient sheaf $W = Y/Z$. Consider the following commutative diagram

$$(45) \quad \begin{array}{ccccc} & & R & \longrightarrow & \mathrm{pt} \\ & & \downarrow & & \downarrow \\ Z \times \mathbb{C} & \longrightarrow & Y \times \mathbb{C} & \longrightarrow & \frac{Y \times \mathbb{C}}{R} \\ \downarrow & & \downarrow & & \\ \mathbb{C} & \longrightarrow & \frac{Y}{Z} \times \mathbb{C} & & \end{array}$$

where the two squares are cocartesian. Consider now the two new cocartesian squares

$$(46) \quad \begin{array}{ccc} Z \times \mathbb{C} & \longrightarrow & \frac{Y \times \mathbb{C}}{R} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & P \end{array} \quad \begin{array}{ccc} R & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \frac{Y}{Z} \times \mathbb{C} & \longrightarrow & \frac{W \times \mathbb{C}}{R} = W \wedge S_{\mathrm{par}}^1. \end{array}$$

By chasing the diagram (45) and using that $\frac{Y}{Z} \times \mathbb{C}$ and $\frac{Y \times \mathbb{C}}{R}$ are colimits of the relevant diagrams, we find two sheaf maps $P \rightarrow W \wedge S_{\mathrm{par}}^1$ and $W \wedge S_{\mathrm{par}}^1 \rightarrow P$ that are mutually inverses. By definition of P we have

$$(47) \quad \begin{aligned} \mathrm{Hom}_{\mathcal{H}_\bullet}(W \wedge S_{\mathrm{par}}^1, X) &= \mathrm{Hom}_{\mathcal{H}_\bullet}(P, X) = \mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}(P, X) = \\ &= \mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}((Y \times \mathbb{C})/R, X) \times_{\mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}(Z \times \mathbb{C}, X)} \mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}(\mathbb{C}, X). \end{aligned}$$

Recall that $(Y \times \mathbb{C})/R = Y \wedge S_{\mathrm{par}}^1$, thus, by the first part of the proof of the proposition, $\mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}((Y \times \mathbb{C})/R, X) = x$, the base point of X . The same holds for $\mathrm{Hom}_{\mathcal{F}_T(S)_\bullet}(\mathbb{C}, X)$ because, by assumption, X is Brody hyperbolic. Therefore,

$$\mathrm{Hom}_{\mathcal{H}_\bullet}(W \wedge S_{\mathrm{par}}^1, X) = x$$

for any quotient sheaf W , and in particular for

$$W = (\mathbb{C} \setminus 0)^{\wedge j} \wedge S_{\text{par}}^{i-j-1}, \quad W = S_{\text{iper}}^n(z_1, z_2) \wedge S_{\text{par}}^{m-1}$$

(see Definition 4.1). \square

Corollary 4.1. *Let \mathcal{X} be a simplicial sheaf. Assume that $\pi_{i,j}^{\text{par}}(\mathcal{X}, x) \neq 0$ for $i-j > 0$ or $\pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X}, x) \neq 0$ for $m > 0$. Then $\mathfrak{Ip}(\mathcal{X})$ is not \mathbb{C} weakly equivalent to a hyperbolic sheaf. In particular, if X is a complex space such that $\pi_{i,j}^{\text{par}}(\mathcal{X}, x) \neq 0$ for $i-j > 0$ or $\pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X}, x) \neq 0$ for $m > 0$, then X is not a Brody hyperbolic complex space.*

Proof. The proofs for the two cases are similar so we consider only the case of the parabolic holotopy groups. By definition,

$$\pi_{i,j}^{\text{par}}(\mathcal{X}, x) = \text{Hom}_{\mathcal{H}_\bullet}((\mathbb{C} \setminus 0)^{\wedge j} \wedge S_s^{i-j}, (\mathcal{X}, x))$$

and this set is a quotient of

$$\text{Hom}_{\Delta_\bullet^{\text{op}} \mathcal{F}_T(\mathcal{S})}((\mathbb{C} \setminus 0)^{\wedge j} \wedge S_s^{i-j}, (\tilde{\mathcal{X}}, \tilde{x}))$$

where $(\tilde{\mathcal{X}}, \tilde{x})$ is an \mathbb{C} -fibrant pointed simplicial sheaf \mathbb{C} -weakly equivalent to (\mathcal{X}, x) . In particular, we may assume that $\tilde{\mathcal{X}}$ is the hyperbolic resolution $\mathfrak{Ip}(\mathcal{X})$ of \mathcal{X} . If $\mathfrak{Ip}(\mathcal{X})$ were \mathbb{C} -weakly equivalent to a Brody hyperbolic complex space X' , then $\pi_{i,j}^{\text{par}}(\mathcal{X}, x)$ would be a quotient of

$$\text{Hom}_{\Delta_\bullet^{\text{op}} \mathcal{F}_T(\mathcal{S})}((\mathbb{C} \setminus 0)^{\wedge j} \wedge S_s^{i-j}, (X', x'))$$

which for $i-j > 0$ consists only in the constant map with value x (cfr. Theorem 4.1). \square

Remark 4.2. As mentioned in section 2, to relate holotopy groups of a complex space X with morphisms in $\Delta^{\text{op}} \mathcal{F}_T(\mathcal{S})$ it is necessary to replace X with its hyperbolic model $\mathfrak{Ip}(X)$. Then we know that $\pi_{i,j}(X, x)$ will be a quotient of the set $\text{Hom}_{\Delta_\bullet^{\text{op}} \mathcal{F}_T(\mathcal{S})}(\mathbf{S}^{i,j}, \mathfrak{Ip}(X))$, where $\mathbf{S}^{i,j}$ is a pointed model of the relevant sphere.

5. THE TOPOLOGICAL REALIZATION FUNCTOR

From now on, \mathbb{CP}^n will denote the complex projective space seen as topological space. We would like to compare objects in \mathcal{H} and $\mathcal{H}(k)$ with the topological spaces, objects of the topological (unstable) homotopy category \mathcal{H}^{top} . We will show that there exists a functor $t^{olo} : \mathcal{H} \rightarrow \mathcal{H}^{top}$ which extends the functor which associates the underlying topological space to a complex space. In the algebraic case extends the corresponding functor which associates to an algebraic variety over \mathbb{C} , the topological space of its (Zariski) closed points. The general case only applies to the site of smooth varieties over a field k which admits an embedding i in \mathbb{C} . It involves passing from a simplicial sheaf over k to a simplicial sheaf over \mathbb{C} by means of i^* (or, more precisely, by means of its total left derived functor). Recall that for a sheaf F and a morphism of sites $\phi : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$, the sheaf $\phi^* F$ on \mathfrak{S}_1 is defined as the associated sheaf to the presheaf whose sections are $(\phi^* F)(U) = \text{colim}_V F(V)$, where the colimit is taken over all the morphisms $U \rightarrow \phi^{-1} V$ for $U \in \mathfrak{S}_1$ and any $V \in \mathfrak{S}_2$.

Definition 5.1. *Let (\mathfrak{S}, I) be a site with interval (cfr. Section 2.3 [16]) equipped with a realization functor $r : \mathfrak{S} \rightarrow \text{Top}$ to the category of topological spaces. Denote by $\mathcal{H}(\mathfrak{S})$ the I homotopy category whose objects are simplicial sheaves over \mathfrak{S} . Then a functor $t_r : \mathcal{H}(\mathfrak{S}) \rightarrow \mathcal{H}^{top}$ with values in the unstable homotopy category of topological spaces is called a topological realization functor if the following properties are satisfied:*

- (1) *if $X \in \Delta^{op}\mathcal{F}(\mathfrak{S})$ is a simplicial set, then the class $t_r(X)$ can be represented by the geometric realization $|X|$;*
- (2) *if F is the sheaf $\text{Hom}_{\mathfrak{S}}(-, X)$, where $X \in \mathfrak{S}$, then $t_r(F)$ can be represented by $r(X)$;*
- (3) *t_r commutes with direct products and homotopy colimits.*

Theorem 5.1. *The sites with interval $(\mathbf{CompI}, \mathbb{C})$ and $((Sm/k)_T, \mathbb{A}_k^1)$ admit a topological realization functor, provided that k can be embedded in \mathbb{C} and T is not finer than the flat topology.*

Proof. Let $\phi : (\mathfrak{S}_1, I_1) \rightarrow (\mathfrak{S}_2, I_2)$ be a reasonable countinuous map of sites with interval (cfr. Definition 1.49 [16]). Consider the functor $\phi^* : \Delta^{op}\mathcal{F}(\mathfrak{S}_2) \rightarrow \Delta^{op}\mathcal{F}(\mathfrak{S}_1)$ obtained by applying the inverse image functor on each component of the simplicial sheaf on \mathfrak{S}_2 . A classical result in model categories assures the existence of the total left derivative between homotopy categories of a functor, provided that such a functor sends weak equivalences between cofibrant objects to weak equivalences. In the case of ϕ^* , we will not be able to prove this for every simplicial sheaf on \mathfrak{S}_2 and the relevant I model categories. However, we can get the same result in the following way. We consider the full category of I_2 local objects $\mathcal{H}_{s, I_2} \subset \mathcal{H}_s$ introduced in the Theorem 2.3, which is equivalent to the I homotopy category $\mathcal{H}(\mathfrak{S}_2, I_2)$ by the same theorem. Such a category has the property that a morphism is an I_2 weak equivalence if and only if it is a simplicial weak equivalence. Thus, to show that ϕ^* admits a total left derived functor between the I homotopy categories, it is sufficient to show that ϕ^* sends simplicial weak equivalences between I_2 local objects (since every object is cofibrant) to simplicial weak equivalences. Actually, since the property for a simplicial sheaf \mathcal{X} to be I_2 local is invariant under simplicial weak equivalences on \mathcal{X} (cfr. Definition 2.3) to validate the same conclusion it suffices to show a weaker condition: there exists a (simplicial) resolution functor Φ and a natural transformation $\Phi \rightarrow \text{id}$ with the property that ϕ^* sends simplicial weak equivalences between simplicial shaves of the kind $\Phi(\mathcal{X}) \rightarrow \Phi(\mathcal{Y})$ to simplicial weak equivalences for all I_2 local simplicial sheaves \mathcal{X} and \mathcal{Y} . But this is precisely the statement of Proposition 1.57.2. of [16] where Φ is taken to be Φ_Σ introduced in Lemma 3.2. This shows the existence of the total left derived functor $\mathbb{L}\phi^* : \mathcal{H}(\mathfrak{S}_2, I_2) \rightarrow \mathcal{H}(\mathfrak{S}_1, I_1)$ of ϕ^* . Explicitely, it is defined as follows: let \mathcal{X} be a simplicial sheaf over \mathfrak{S}_2 , then $\mathbb{L}\phi^*(\mathcal{X})$ is represented by the simplicial sheaf $\phi^*(\Phi_\Sigma(\mathfrak{I}\mathfrak{p}(\mathcal{X})))$, where $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ is the I_2 local simplicial sheaf mentioned in the Theorem 2.3. This definition is well posed on $\mathcal{H}(\mathfrak{S}_2, I_2)$ because of the above remarks

and the fact that, if \mathcal{X} and \mathcal{X}' represent the same class in $\mathcal{H}(\mathfrak{S}_2, I_2)$, then $\mathfrak{I}\mathfrak{p}(\mathcal{X})$ and $\mathfrak{I}\mathfrak{p}(\mathcal{X}')$ are simplicially weak equivalent.

We will now consider the case of the site with interval $(\mathbf{CompI}, \mathbb{C})$ since the algebraic case when $k = \mathbb{C}$ is entirely similar. We set the realization functor $r : \mathfrak{S} \rightarrow Top$ to be the one which associates the underlying topological space X^{top} to a complex space X . Let \mathbf{pt} be the site with interval whose only nonempty object is the final object \mathbf{pt} and ψ be the trivial morphism of sites with interval $\mathbf{pt} \rightarrow \mathbf{CompI}$. Notice that a simplicial sheaf on \mathbf{pt} is just a simplicial set. We take the interval I in \mathbf{pt} to be the constant simplicial set $\mathbf{Hom}(\mathbf{pt}, \mathbb{C})$. The functor ψ^* sends a simplicial sheaf \mathcal{X} on \mathbf{CompI} to the simplicial set $\mathcal{X}(\mathbf{pt})$. Thus, $\psi^*(\mathbb{C}) = I$ so that, in particular, it is I contractible. Because of this, the functor ψ is said to be a *reasonable* continuous map of sites with interval (cfr. Definition 3.16, [16]) and $\mathbb{L}\psi^*$ has a particularly nice description: $\mathbb{L}\psi^*(\mathcal{X})$ is represented by the simplicial sheaf $\psi^*\Phi_\Sigma(\mathcal{X})$ where Σ is the class of representable sheaves on \mathbf{CompI} (see Lemma 3.15. of [16]). Let Tlc_{open} be the category of locally contractible topological spaces. We now endow the images of $\mathbb{L}\psi^*$ by a structure of topological spaces in order to obtain a functor $\mathcal{H}(\Delta^{op}\mathcal{F}(\mathbf{CompI})) \rightarrow \mathcal{H}(\Delta^{op}Tlc_{open})$. If \mathcal{Y} is a simplicial sheaf that in each degree is a disjoint union of representable sheaves $\coprod_{j \in J} Y_j$, then we set $\theta\mathcal{Y} := \mathcal{Y}^{top}$, where \mathcal{Y}^{top} is the simplicial topological space having the topological space $\coprod_{j \in J} \mathcal{Y}_j^{top}$ in the corresponding degree. Since ψ^* is reasonable, $\mathbb{L}\psi^*(\mathcal{X}) = [\psi^*\mathcal{Y}]$ where \mathcal{Y} is any representable simplicial sheaf equipped with a simplicial weak equivalence $\mathcal{Y} \rightarrow \mathcal{X}$. Any two such models will give rise to simplicially weak equivalent inverse images by Proposition 1.57.2. [16], thus, in particular, I weak equivalent. This shows that the definition of θ induces a functor

$$\mathcal{H} = \mathcal{H}(\Delta^{op}\mathcal{F}(\mathbf{CompI})) \rightarrow \mathcal{H}(\Delta^{op}Tlc_{open})$$

which we will call θ , as well.

Remark 5.1. $\mathcal{H}(\Delta^{op}Tlc_{open})$ is a full subcategory of $\mathcal{H}(\Delta^{op}\mathcal{F}_{open}(Tlc_{open}))$. The latter category is the I homotopy category taking as interval the sheaf $I = \mathbf{Hom}_{cont}(\cdot, \mathbb{C})$. Such an interval is an object of $\Delta^{op}Tlc_{open}$, thus we can see $\mathcal{H}(\Delta^{op}Tlc_{open})$ as the

localized category with respect to the $I = \mathbb{C}$ weak equivalences, considering \mathbb{C} as constant simplicial topological space and no longer only as constant simplicial set.

Proposition 5.1. *There is an equivalence of categories $\gamma : \mathcal{H}(\Delta^{op}Tlc_{open}) \cong \mathcal{H}^{top}$.*

Proof. (sketch) It is a particular case of Proposition 3.3 of [16]. Here we write the definition of the functor $\gamma : \mathcal{H}(\Delta^{op}Tlc_{open}) \rightarrow \mathcal{H}^{top}$ which gives the equivalence of categories. Let \mathcal{X} be a simplicial locally contractible topological space. Since for any topological space Z in Tlc_{open} there is an open covering $\coprod_i U_i \rightarrow Z$, with U_i contractible for all i , by Lemma 3.2, \mathcal{X} admits a (simplicial) weak equivalence $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ with $\tilde{\mathcal{X}}_j = \coprod_{i_j} U_{i_j}$. In turn, $\tilde{\mathcal{X}}$ is I weakly equivalent to \mathcal{X}' , where \mathcal{X}' is the *simplicial set* with $\mathcal{X}'_j = \coprod_{i_j} \mathbf{pt}$, because $\tilde{\mathcal{X}}$ and \mathcal{X}' are termwise weakly equivalent and of Proposition 2.14, [16]. The equivalence of categories is defined as $[\mathcal{X}] \rightsquigarrow [|\mathcal{X}'|]$ where $|\mathcal{X}'|$ is the geometric realization of the simplicial set \mathcal{X}' . \square

Remark 5.2. If X is a topological space in Tlc_{open} , then $|\mathcal{X}'|$ is weakly equivalent to $|Sing_{\bullet}(X)|$. But this topological space is weakly equivalent to X itself, thus the constant simplicial topological space X is sent by γ to a topological space weakly equivalent to X in the classical sense of homotopy theory.

Let \mathbf{D} be a small category and $\Delta^{op}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ be the category of functors from \mathbf{D} to $\Delta^{op}\mathcal{F}_T(\mathcal{S})$. We will denote by $\mathrm{hocolim}(\mathbf{D})$ a *homotopy colimit* of \mathbf{D} on the category $\Delta^{op}\mathcal{F}_T(\mathcal{S})$. That is a pair (k, a) consisting in a functor $k : \Delta^{op}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}} \rightarrow \Delta^{op}\mathcal{F}_T(\mathcal{S})$ which takes objectwise weak equivalences in $\Delta^{op}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ to weak equivalences in $\Delta^{op}\mathcal{F}_T(\mathcal{S})$ and a natural transformation $a : k \rightarrow \mathrm{colim}_{\mathbf{D}}$. Such functor can be obtained by first taking a suitable cofibrant diagram replacement of an element in $\Delta^{op}\mathcal{F}_T(\mathcal{S})^{\mathbf{D}}$ and composed with the ordinary colimit functor (cfr. [2]).

We set $t^{olo} : \mathcal{H} \rightarrow \mathcal{H}^{top}$ to be the functor $\gamma \circ \theta$. Property (1) of Definition 5.1 follows by definition of γ . Property (2) is a consequence of Remark 5.2. As for the property (3), we have that ψ^* commutes with limits by definition. Since direct products in the homotopy categories are represented by direct products of

objects, we have that t^{olo} commutes with direct products. ψ^* has a right adjoint, namely ψ_* , thus it is right exact. Moreover, ψ^* sends cofibrations (sectionwise injections) to cofibrations. On the other hand, the same holds for the resolution functor of Lemma 3.2 Φ : if i is a sectionwise injection, then $\Phi(i)$ is a sectionwise injection by definition of Φ ; furthermore, Φ commutes with colimits, since its value on objects has been defined as a colimit. In particular, if \mathbf{D} is a cofibrant diagram in $\Delta^{op}\mathcal{F}(\mathcal{C}\mathbf{ompl})$, $\Phi(\mathbf{D})$ is cofibrant and we conclude that $\psi^*\Phi(\mathbf{D})$ is cofibrant as well and also that $\text{colim}(\psi^*\Phi(\mathbf{D})) \cong \psi^*\Phi^*(\text{colim}(\mathbf{D}))$. This shows that, for any diagram \mathbf{D} , $\mathbb{L}\psi^*(\text{hocolim}(\mathbf{D})) = \text{hocolim}(\mathbb{L}\psi^*(\mathbf{D}))$, since the former class can be represented by $\text{colim}(\psi^*\Phi(\mathbf{D}'))$ for any cofibrant replacement $\mathbf{D}' \xrightarrow{\sim} \mathbf{D}$ because $\psi^*\Phi(\mathbf{D}')$ is a cofibrant diagram.

Therefore, θ commutes with homotopy colimits. Recall that the equivalence γ is defined to be the functor that, to a class represented by a simplicial topological space \mathcal{X} , associates the class in \mathcal{H}^{top} represented by $|(\Phi_S(\mathcal{X}))^\sim|$ where S is the class of contractible topological spaces and the operation \sim replaces each contractible topological space with a point. Because of the definition of Φ_S we see that \sim sends injections to injections and commutes with colimits. Before proceeding to investigate the properties of the functor $| |$, we need to recall the model structures involved in the categories. The functor $| |$ is defined on the category of simplicial sets and takes values in the category of topological spaces. The model structure for the category of simplicial sets is: let $f : X \rightarrow Y$ be a map of simplicial sets, then f is:

- (1) a weak equivalence if $|f|$ is a weak homotopy equivalence (see below);
- (2) a cofibration if it is an injection;
- (3) a fibration if f has the right lifting property with respect to acyclic cofibrations.

Let $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \rightarrow \dots$ be a sequential direct system of topological spaces such that for each n , (X_n, X_{n+1}) is a relative CW complex. Then we will say that

the canonical function $X_0 \hookrightarrow \operatorname{colim} X_i$ is a *generalized relative CW inclusion*. A continuous function between topological spaces $f : X \rightarrow Y$ is

- (1) *a weak equivalence if $f_* : \pi_*(X, x) \rightarrow \pi_*(Y, f(x))$ is a group isomorphism for $* \geq 1$ and a bijection of pointed sets if $* = 0$;*
- (2) *a cofibration if it is a retract of a generalized relative CW inclusion;*
- (3) *a fibration if it is a Serre fibration.*

The functor $|\cdot|$ preserves cofibrations and also it commutes with colimits, because it has a right adjoint, namely the functor $\operatorname{Sing}(\cdot)$. In conclusion, the functor γ commutes with homotopy colimits, and so does the topological realization functor t^{olo} .

□

5.1. Remarks on homotopy colimits. The practical use of the topological realization functor requires few remarks on the differences between (homotopy) colimits of diagrams in the category \mathcal{H}^{top} and the category \mathcal{H} . Let us consider the colimit of the diagram

$$(48) \quad \begin{array}{ccc} \mathbb{C} \setminus 0 & \hookrightarrow & \mathbb{C} \\ \downarrow & & \\ \text{pt.} & & \end{array}$$

In the category of complex spaces, this is just a point. However, we have previously inferred in this manuscript that the colimit of such a diagram in the category of sheaves on \mathbf{CompI} is not (weakly equivalent to) the constant sheaf to a point. Indeed, its class in the respective homotopy categories plays the role of the two dimensional sphere $S^2 = \mathbb{CP}^1$, or, more precisely, of the sheaf represented by \mathbb{CP}^1 , whose class is by no means isomorphic to the one of the point. As a diagram of topological spaces, its colimit is not a point, but it is not an appropriate model for S^2 . We should point out that the diagram (48) is a cofibrant diagram for the affine model structure in the category $\Delta^{op}\mathcal{F}(\mathbf{CompI})$, but it is not cofibrant in the category of topological spaces for the model structure defined above. This apparent oddness

disappears if we consider *homotopy* colimits instead. For instance,

$$t^{olo}(S_{\text{par}}^n) \cong t^{olo}(S_{\text{iper}}^n(z_1, z_2)) \cong S^n$$

for any $n \geq 0$ and $z_i \in D$, $t^{olo}(\mathbb{C} \setminus 0) \cong S^1$ and we have natural maps of pointed sets (respectively of groups)

$$\pi_{i,j}^{\text{par}}(\mathcal{X}, x) \rightarrow \pi_i(t^{olo}(\mathcal{X}), t^{olo}(x))$$

and

$$\pi_{n,m}^{\text{iper}}(\mathcal{X}, x) \rightarrow \pi_{n+m}(t(\mathcal{X}), t^{olo}(x)).$$

6. SOME APPLICATIONS

In this last section we are going to consider few applications of the theory developed so far. We will begin with examples of complex spaces that are not \mathbb{C} weakly equivalent to any complex hyperbolic space.

Definition 6.1. *We will say that a complex space is weakly hyperbolic if it is \mathbb{C} weakly equivalent to a Brody hyperbolic complex space.*

We recall a preliminary result (cfr. Lemma 2.15 [16]):

Lemma 6.1. *The pointed simplicial sheaf $(\mathbb{C} \setminus 0) \wedge S_{\text{par}}^1$ is canonically weakly equivalent to \mathbb{P}^1 .*

Proof. Consider the diagram **D**

$$(49) \quad \begin{array}{ccc} (\mathbb{C} \setminus \{0\}, \{1\}) & \longrightarrow & (\mathbb{C}, \{1\}) \\ \downarrow & & \\ (\mathbb{C} \setminus \{0\}, \{1\}) \wedge \Delta[1]. & & \end{array}$$

If **D'** is another diagram

$$(50) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & Y \\ \downarrow i & & \\ \mathcal{Z} & & \end{array}$$

in \mathcal{H} then $\text{colim}_{\mathbf{D}} \cong \text{colim}_{\mathbf{D}'}$ in \mathcal{H} if there exists a morphism of diagrams $\mathbf{D} \rightarrow \mathbf{D}'$ such that the morphisms are weak affine equivalences. Consider the diagrams **D'** e

\mathbf{D}''

$$(51) \quad \begin{array}{ccc} (\mathbb{C} \setminus \{0\}, \{1\}) & \longrightarrow & \mathbf{pt} \\ \downarrow & & \downarrow \\ (\mathbb{C} \setminus \{0\}, \{1\}) \wedge \Delta[1] & & \mathbf{pt} \end{array} \quad \begin{array}{ccc} (\mathbb{C} \setminus \{0\}, \{1\}) & \longrightarrow & (\mathbb{C}, \{1\}) \\ \downarrow & & \downarrow \\ \mathbf{pt} & & \mathbf{pt} \end{array}$$

and the morphisms

$$f = ((\mathbb{C}, \{1\}) \rightarrow \mathbf{pt}, \text{id}) : \mathbf{D} \rightarrow \mathbf{D}' \quad g = (\text{id}, (\mathbb{C} \setminus \{0\}, \{1\}) \wedge \Delta[1] \rightarrow \mathbf{pt}) : \mathbf{D} \rightarrow \mathbf{D}''.$$

The morphisms f and g induce affine weak equivalences $\text{colim}_{\mathbf{D}} \rightarrow \text{colim}_{\mathbf{D}'}$ and $\text{colim}_{\mathbf{D}} \rightarrow \text{colim}_{\mathbf{D}''}$. Identifying $\text{colim}_{\mathbf{D}'}$ with $(\mathbb{C} \setminus \{0\}, \{1\}) \wedge S_s^1$ and $\text{colim}_{\mathbf{D}''}$ with $\mathbb{C}/(\mathbb{C} \setminus \{0\})$, we conclude that

$$(\mathbb{C} \setminus \{0\}, \{1\}) \wedge S_s^1 \cong \mathbb{C}/(\mathbb{C} \setminus \{0\}).$$

The square

$$(52) \quad \begin{array}{ccc} \mathbb{C} \setminus \{0\} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \setminus \{\infty\} & \longrightarrow & \mathbb{P}^1 \end{array}$$

is cocartesian in $\mathcal{F}_T(\mathcal{S})$, hence the cofibres of horizontal morphisms are isomorphic.

We derive

$$\mathbb{C}/(\mathbb{C} \setminus \{0\}) \cong \mathbb{P}^1/(\mathbb{P}^1 \setminus \{\infty\})$$

in $\mathcal{F}_T(\mathcal{S})$. But

$$\mathbb{P}^1/(\mathbb{P}^1 \setminus \{\infty\}) \cong \mathbb{P}^1$$

in \mathcal{H} , since $\mathbb{P}^1 \setminus \{\infty\} \cong \mathbf{pt}$ in \mathcal{H} .

Remark 6.1. In the proof of Lemma 4.1 we have already seen that S_{par}^1 is weakly equivalent to S_s^1 .

We are now going to apply the theory developed so far to prove that

Theorem 6.1. *For any $n > 0$, \mathbb{P}^n is not weakly hyperbolic. In other words, $\mathfrak{I}\mathfrak{p}(\mathbb{P}^n)$ cannot be represented in \mathcal{H} by a Brody hyperbolic complex space.*

Proof. In view of Corollary 4.1, it is sufficient to show that

$$\pi_{2,1}^{\text{par}}(\mathbb{P}^n, \infty) = \text{Hom}_{\mathcal{H}_\bullet}((\mathbb{C} \setminus \{0\}) \wedge S_{\text{par}}^1(\mathbb{P}^n, \{\infty\})) \neq 0$$

or equivalently, by Lemma 6.1 and Remark 6.1, that

$$\text{Hom}_{\mathcal{H}_\bullet}(\mathbb{P}^1, (\mathbb{P}^n, \{\infty\})) \neq 0.$$

Our candidate to represent a nonzero class is the canonical embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$.

The topological realization yields a group homomorphism

$$t : \pi_{2,1}^{\text{par}}(\mathbb{P}^n, \infty) \rightarrow \pi_2(\mathbb{CP}^n, \infty).$$

$t^{olo}(i) : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ is the canonical inclusion and not null homotopic, since \mathbb{CP}^n is obtained by \mathbb{CP}^1 by attaching cells of dimension 4 and above, hence it is an equivalence up to dimension 2 and in particular

$$t^{olo}(i)_* : \mathbb{Z} = \pi_2(\mathbb{CP}^1, \infty) \rightarrow \pi_2(\mathbb{CP}^n, \infty)$$

is an isomorphism. In conclusion $t[i] \neq 0$, thus $[i] \neq 0 \in \pi_{2,1}^{\text{par}}(\mathbb{P}^n, \infty)$. \square

Proposition 6.1. *Let X be a complex space and $p : \tilde{X} \rightarrow X$ a connected covering complex space. Assume that X is weakly hyperbolic and let $f : \mathbb{C} \rightarrow X$ be a nonconstant holomorphic function. Then for any lifting \tilde{f} of f to \tilde{X} , $\tilde{f}(\mathbb{C})$ contains just one point in each fiber of p or equivalently $p|_{\tilde{f}(\mathbb{C})}$ is a biholomorphism for any such f and \tilde{f} .*

Proof. Let X be weakly hyperbolic. Assume, by a contradiction, that there exist a nonconstant holomorphic function $f : \mathbb{C} \rightarrow X$ and a lifting $\tilde{f} : \mathbb{C} \rightarrow \tilde{X}$ such that $a \neq b \in p^{-1}(x)$, $x \in X$, $a, b \in \tilde{f}(\mathbb{C})$. For the purposes of this proof, we can assume that $\tilde{f}(0) = a$ and $\tilde{f}(1) = b$. Then we have the following commutative diagram:

$$(53) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow q & & \downarrow p \\ \mathbb{C}/\{0\} \amalg \{1\} & \xrightarrow{\alpha} & X \end{array}$$

where α sends the class of $\{0\} \amalg \{1\}$ to $x \in X$. We have that $[\alpha] \neq 0 \in \pi_{1,0}^{\text{par}}(X, x)$. Indeed, $[\alpha^{top}] \neq 0 \in \pi_1(X^{top}, x)$. Consider the composition

$$[0, 1] \xrightarrow{g} \mathbb{C}/\{0\} \amalg \{1\} \xrightarrow{\alpha^{top}} X^{top},$$

where g is a path from 0 to 1 in \mathbb{C} . If $\alpha^{top} \circ g$ is not homotopic to a constant relatively to $\{0, 1\}$, then α^{top} is not homotopic to a constant. But, by construction, $\alpha^{top} \circ g$ lifts uniquely to a path in \tilde{X}^{top} starting from a and ending in b , hence $\alpha^{top} \circ g$ cannot be homotopic to a constant relatively to $\{0, 1\}$. This shows that $\pi_1(X^{top}, x) \neq 0$ which is absurd since X is weak hyperbolic. \square

The Proposition 6.1 in particular implies the following

Corollary 6.1. *Any complex space X whose universal covering space is \mathbb{C}^n for some $n \geq 1$, is not weakly hyperbolic.*

Proof. Let $p : \mathbb{C}^n \rightarrow X$ be the universal covering of X . Let $a \neq b \in p^{-1}(x)$, $x \in X$. A complex line $l \subset \mathbb{C}^n$ passing through a, b provides a homomorphic map $f : \mathbb{C} \rightarrow X$ which does not satisfy the conclusion of Proposition 6.1. \square

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